PERGAMON

Omega, Int. J. Mgmt. Sci. 27 (1999) 537–553

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The International Journal of Management Science

www.elsevier.com/locate/orms

# The single-period (news-vendor) problem: literature review and suggestions for future research

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Received 1 April 1998; received in revised form 1 January 1999; accepted 1 January 1999

**Abstract**

The single-period problem (SPP), also known as the newsboy or news-vendor problem, is to find the order quantity which maximizes the expected profit in a single period probabilistic demand framework. Interest in the SPP remains unabated and many extensions to it have been proposed in the last decade. These extensions include dealing with different objectives and utility functions, different supplier pricing policies, different news-vendor pricing policies and discounting structures, different states of information about demand, constrained multi-products, multiple-products with substitution, random yields, and multi-location models. This paper builds a taxonomy of the SPP literature and delineates the contribution of the different SPP extensions. This paper also suggests some future directions for research. 

*Keywords:* Inventory management and production/operations management

## 1. Introduction

The classical single-period problem (SPP) is to find a product's order quantity that maximizes the expected profit under probabilistic demand. The SPP model assumes that if any inventory remains at the end of the period, a discount is used to sell it or it is disposed of [1]. If the order quantity is smaller than the realized demand, the news-vendor, hereafter NV, forgoes some profit. The SPP is reflective of many real life situations and is often used to aid decision making in the fashion and sporting industries, both at the manufacturing and retail levels [2]. The SPP can also be used in managing capacity and evaluating advanced booking of orders in service industries such as airlines and hotels [3].

Interest in the SPP has increased in the last decade with over 40 papers published since 1988. In this paper we review the literature on the SPP. The SPP literature is very large and complete coverage is beyond the scope of a single paper. A partial review of the SPP literature has been recently conducted in a textbook by Silver et al. [4]. Because of the depth of the SPP literature, many of the papers we review here were omitted in that review. To avoid redundancy we concentrate our efforts on papers that received little or no treatment in Silver et al.'s book. In Section 2 we introduce the SPP. In Section 3 we develop a taxonomy of SPP extensions and place the reviewed models into the classes of the taxonomy. In Section 4 we provide a discussion of the models. We close with some concluding remarks and suggestions for future research in Section 5.

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0305-0483/99/$ - see front matter 
PII: S0305-0483(99)00017-1

## 2. The classical single-period problem

Researchers have followed two approaches to solving the SPP. In the first approach, the expected costs of overestimating and underestimating demand are minimized. In the second approach, the expected profit is maximized. Both approaches yield the same results. We use the second approach in stating the SPP. Define the following notation:

| | |
|---|---|
| $x$ | quantity demanded, a random variable. |
| $f(x)$ | the probability density function of $x$. |
| $F(x)$ | the cumulative distribution function of $x$. |
| $P$ | selling price per unit. |
| $C$ | cost per unit. |
| $V$ | salvage value per unit. |
| $S$ | shortage penalty cost per unit. |
| $C_o$ | $C-V$, unit overage cost. |
| $C_u$ | $P-C+S$, unit underage cost. |
| $Q$ | order quantity, a decision variable. |

The profit per period is

$$\pi = \begin{cases} (P-C) \quad Q - S(x-Q), & \text{if } x \geq Q \\ Px + V \quad (Q-x) - CQ, & \text{if } x < Q. \end{cases} \tag{1}$$

Simplifying and taking the expected value of $\pi$ gives the following expected profit

$$E(\pi) = (P+S-C)\int_Q^\infty Qf(x)\,\mathrm{d}x - S\int_Q^\infty xf(x)\,\mathrm{d}x + (P-V)\int_0^Q xf(x)\,\mathrm{d}x - (C-V)\int_0^Q Qf(x)\,\mathrm{d}x \tag{2}$$

Let the superscript * denote optimality. Using Leibniz's rule to obtain the first and second derivatives shows that $E(\pi)$ is concave. The sufficient optimality condition is the well known fractile formula:

$$F(Q^*) = \frac{P+S-C}{P+S-V} \tag{3}$$

Identical results can be obtained by minimizing the expected underage and overage costs. Many authors describe the overage cost ($C_o$) as a cost of holding inventory which is charged to the ending inventory.

## 3. Extensions of the classical single-period model

The SPP has wide applicability especially in service industries which dominates the US economy. As product life cycles continue their downward trend, the importance of the SPP will grow. It is not surprising that many SPP extensions have been suggested with many of them appearing in the last five years. Extensions to the SPP can be classified into 11 categories:

1. Extensions to different objectives and utility functions.
2. Extensions to different supplier pricing policies.
3. Extensions to different news-vendor pricing policies and discounting structures.
4. Extensions to random yields.
5. Extensions to different states of information about demand.
6. Extensions to constrained multi-product.
7. Extensions to multi-product with substitution.
8. Extensions to multi-echelon systems.
9. Extensions to multi-location models.
10. Extensions to models with more than one period to prepare for the selling season.
11. Other extensions.

It should be noted that a paper may fall into more than one of the 11 categories shown above. In that case, the paper is placed in the category of its dominant contribution.

### *3.1. Extensions to different objectives and utility functions*

Researchers observed that maximizing $E(\pi)$ may not reflect reality. Actually, maximizing the probability of achieving a target profit was empirically found to be more consistent with the actions of many managers [5]. Subsequently, researchers proposed extensions to the SPP in which the goal is to maximize the probability of achieving a target profit [6–14]. Other authors used different effectiveness criterion, risk tolerance and utility functions [15–21].

Kabak and Schiff [7] solved the SPP under the 'satisficing' objective of maximizing the probability of achieving a target profit of $B$, denoted $P_B$. Kabak and Schiff derived the necessary condition for $Q^*$ and provided a closed-form solution for exponentially distributed demand.

Work on a variation of the SPP was also being carried out in the accounting literature under the cost–volume–profit ($C$–$V$–$P$) analysis. The $C$–$V$–$P$ states:

$$\text{Total Profit} = \text{Sales Volume} \times (\text{Unit Selling Price} - \text{Unit Variable Cost}) - \text{Fixed Cost}. \tag{4}$$

Shih [14] observed that a deficiency in the stochastic $C$–$V$–$P$ is that even though demand is treated as a random variable, the effects of any unsold units on profit was not taken into account. Shih considered the effects of over production and derived a general probability distribution of $\pi$, its expected value, and its variance as a function of $Q$. For normally distributed demand, Shih derived the probability distribution, the mean and variance of $\pi$. Also, Shih derived an expression for $Q$ which maximizes $P_B$. Finley and Liao [22] claimed that Shih's analysis may be erroneous and proposed an analysis which fixes the error. Lau and Lau [23] provided simpler formulas for computing the mean and variance of $\pi$ and the order quantity which maximizes $P_B$.

Ismail and Louderback [6] also used the $C$–$V$–$P$ framework and incorporated a per unit shortage cost. Ismail and Louderback showed that under shortage penalties, profits will not necessarily conform to any well-defined distribution. Ismail and Louderback derived the necessary condition for $Q$ to maximize $E(\pi)$, which is the classical result in Eq. (3). Ismail and Louderback showed that for normally distributed demand, the distribution of profit can be far from the normal distribution and is not even symmetrical. Under these conditions, the mean-variance method for evaluating risk is inappropriate. Therefore, Ismail and Louderback suggested a new objective to treat the risk-return tradeoff, which is maximizing $P_B$ as proposed by Kabak and Schiff [7]. Ismail and Louderback then provided an iterative procedure for determining $Q^*$. Ismail and Louderback also used the objective of maximizing $P_B$, given a target level of the probability of their being achieved and developed an iterative procedure for finding $Q^*$.

Lau [8] revisited Ismail and Louderback's [6] model and provided convenient procedures for computing $E(\pi)$ and $P_B$ for normal, Beta, and Gamma distributed demand. Lau also developed a simple and exact analytical solution for $Q^*$ which maximizes $P_B$.

Norland [12] identified maximizing $P_B$ as the aspiration criterion and maximizing $E(\pi)$, given a target level of probability of their being achieved, as the fractile criterion as they are known in the literature. Like Lau [8], Norland derived an analytical expression for $Q^*$ for the aspiration criterion under normal demand. He also improved Ismail and Louderback's procedure for identifying $Q^*$ for the fractile criterion by exploiting the duality between the aspiration and fractile criteria. For a shortage penalty cost of $S = 0$, Norland obtained analytic expressions for $Q^*$ that are valid for any demand distribution for both aspiration and fractile criteria.

Lau [9] provided more detailed analysis under the objective of maximizing $P_B$ and added the objective of maximizing the expected utility. For maximizing $P_B$, Lau addressed two cases: a shortage cost of $S = 0$ and $S > 0$. For $S = 0$, $Q^*$ is independent of the demand distribution and is given by $Q^* = B/(P-C)$. For $S > 0$, $Q^*$ depends on the demand distribution. Lau provided first order conditions and obtained closed-form expressions for $Q^*$ under normal and Schmeiser–Deutsch demand distributions. Numerical methods will have to be used for the Beta, Gamma, and Weibull distributions. Lau then considered the mean-standard deviation of profit tradeoff using $u(\pi) = E(\pi) - k\sigma(\pi)$, where $u(\pi)$ is the utility of profit, $\sigma(\pi)$ is its standard deviation and $k$'s magnitude reflects an NV's degree of risk aversion. $Q^*$ has to be numerically evaluated. Lau also maximized the von Neumann–Morgenstern's expected utility of the NV resulting in a $Q^*$ which has to be obtained numerically. Sankarasubramanian and Kumaraswamy [13] also maximized $P_B$. Sankarasubramanian and Kumaraswamy provided closed-form solutions for $Q^*$ for exponential and uniform demand distributions. Sankarasubramanian and Kumaraswamy also solved the case in which the commodity sold is a luxury item and demand is proportional to income.

Lau and Lau [10] revisited the SPP under the objective of maximizing $P_B$ for the two-product case. Lau and Lau considered $S = 0$ and $S > 0$ separately because the latter is more complex. Lau and Lau identified three approaches to estimating $P_B$ and $Q_i^*$, $i = 1$, 2: (1) simulate the problem, (2) develop an expression for $P_B$ and find $Q_i^*$, $i = 1$, 2 using a 'hill-climbing' procedure and (3) analytically solve the first order conditions. Approach 2 was found to be the only practical one. Lau and Lau provided numerical solution to a two-product SPP with uniform and normal demands and provided some insights into management behavior. Lau and Lau then derived general expressions for $P_B$ and closed-form expressions for it under uniform demand distributions. Lau and Lau derived $Q_i^*$, $i = 1$, 2 for two identical products. Lau and Lau found some counter-intuitive results. For example, if a firm has two single-product divisions and each division will receive a bonus for achieving a certain profit, it is beneficial for the divisions to cooperate if the targets are lax and profit margins are high, but not if the targets are high and margins are low.

Driven by the desire to explain some of the results of Lau and Lau [10], Li et al. [11] provided analytical results for the two-product SPP with the objective of maximizing $P_B$ and independent exponential demand distributions. Li et al. developed a procedure for determining $Q_i^*$, $i = 1$, 2, verified their results using simulation and were also better able to explain some of the earlier interesting results of Lau and Lau [10].

Atkinson [16] analyzed the effects of incentives in the SPP. In this model, the owner hires a manager because by virtue of specialization the manager can make better decisions. Atkinson analyzed the effects of different incentive systems on the decision of the manager and the income of the owner and manager. Atkinson analyzed the effect of using a standard-setting mechanism in the reward structure resulting in the manager's monetary return being:

$$\text{Managers Return} = \text{Wage} + k \times (\text{Profit due to manager's decision} - \text{Profit due to standard quantity}), \quad (5)$$

where $k$ is the manager's share of differential return and the standard quantity is the quantity set by the owner based on historical data. When Eq. (5) is not used, Atkinson showed that a risk-averse manager will order a quantity smaller than a risk-neutral NV and that Eq. (5) can be used to mitigate the manager's risk. If the manager deviates from the standard set by the owner, it is because the manager's assessment of demand differs from the owner. Atkinson also showed that under certain conditions and using the standard-setting mechanism, if the owner delegates the choice to the manager, then the owner's position is usually improved. Atkinson identified the manipulation of internal prices of resources transferred within the organization as means of changing the manager's behavior, but cautioned that this scheme is unpredictable and may be cost-ineffective.

Thakkar et al. [20] argued that the maximizing $E(\pi)$ or $P_B$ criteria do not consider the investment that must be made to attain that profit and used return on investment (ROI) as a criteria. Thakkar et al. solved the SPP under two objectives: (a) maximizing $E$(ROI) and (b) maximizing the probability of achieving a target ROI, $P_{\text{ROI}}$. Using incremental analysis to maximize $E$(ROI), Thakkar et al. showed there is a unique $Q^*$ and derived the necessary optimality condition. Thakkar et al. simplified the necessary condition for the normal distribution to an equation that can be manually solved. For maximizing $P_{\text{ROI}}$, Thakkar et al. showed that a simple iterative procedure can be used to solve the discrete demand case and a search procedure can be used for continuous demand and obtained a closed-form solution for normally distributed demand.

Magee [19] and later Anvari [15] employed the capital-asset pricing model (CAPM) for the SPP. Magee [19] suggested that the true measure of risk for the NV is not the variance of profit but rather the covariance of profit with the return on a market portfolio of securities. Consequently, Magee used the CAPM framework to solve the SPP. Anvari [15] also provided an argument on how risk will effect $Q^*$. Similar to Magee's model, this model captures the notion that rational individuals will hold multiple-asset portfolios and thus will be concerned with the covariance risk. Anvari derived the optimality condition and developed an algorithm for finding $Q^*$. Thorstenson [21] claimed that Anvari's assumption that the total capital to be invested by the firm is fixed leads to a case of capital rationing which is not consistent with the perfect capital-market assumption of the CAPM. Chung [17] simplified Anvari's optimality conditions and provided a simple method for computing $Q^*$.

Eeckhoudt et al. [18] examined various comparative statics for the risk-averse NV. Eeckhoudt et al. considered the effects of two types of increase in risk on $Q^*$: (a) the addition of an independent risk to the NV's background wealth and (b) an increase in the riskiness of newspaper demand. Eeckhoudt et al. assumed the NV's preference functional over final wealth distributions is of the expected-utility type, with $u(.)$ denoting the utility of wealth. Eeckhoudt et al. also assumed that the NV is weakly risk-averse (i.e. either risk-averse or risk-neutral). Thus $u$ is increasing and concave. Eeckhoudt et al. conclusions included: (1) the risk-averse NV orders fewer newspapers than the risk-neutral NV, (2) if the NV's preferences exhibit the commonly assumed property of decreasing absolute risk aversion, then wealthier NVs will order more newspapers, (3) $Q^*$ will increase if the salvage value of the newspaper increases and (4) $Q^*$ will decrease if the cost of the newspaper increases. Eeckhoudt et al. provided detailed analysis of the qualitative effects of changes in demand risk on $Q^*$.

### 3.2. Extensions to different supplier pricing policies

The determination of $Q^*$ when suppliers offer quantity discounts has been subject of many SPP extensions [24–28]. Jucker and Rosenblatt [24] considered three types of quantity discounts:

1. All-units quantity discounts. The supplier has price schedule with price breaks at quantities:

   $$0 = q_0 < q_1 < q_2 < \cdots < q_j < \cdots < q_m = \infty$$

   For order quantity $Q$ such that $q_j < Q < q_{j+1}$, the cost per unit is $C_j$ and $C_0 > C_1 > C_2 > \cdots > C_j > \cdots > C_m$. In other words, the discount applies to all units purchased.
2. Incremental quantity discounts. The discount applies only to the additional units after the breakpoint.
3. Carload-lot discounts in which a flat rate of $t$ is charged for each unit shipped, determined by the weight, $w$, of the unit up to some fraction $\alpha$ of a car's weight capacity $L$. Any quantity such that $\alpha L \leq wQ \leq L$ is considered as a 'carload-lot' and assessed the maximum total cost.

Jucker and Rosenblatt showed that the behavior of an NV facing an all-units quantity discount depends on the cost of disposing of excess inventory which can be: (i) zero, (ii) negative and (iii) positive (items have no salvage value and are costly to dispose of). Also, Jucker and Rosenblatt identified two types of supplier behavior: (a) 'cooperative' in which the supplier allows the NV to take delivery of only $Q$ units even if the actual order quantity is at the breakpoint just above $Q$ and (b) 'literal' in which the NV takes delivery of the whole quantity ordered. Analysis of the behavior implications under the all-units quantity discount resulted in a marginal cost function that has intervals in the discount schedule that effectively have zero marginal cost. This marginal cost function is general enough to admit the marginal cost functions of the incremental and carload-lot schedules as special cases. The marginal cost function allowed Jucker and Rosenblatt to develop a solution procedure simpler and more efficient than traditional approaches.

Pantumsinchai and Knowles [28] proposed algorithms for solving an SPP in which $Q$ is made up of a number of containers with standard sizes. The NV can choose any combination of container sizes. The larger the container the smaller the unit cost. A restricted policy is one in which $Q$ is filled by first ordering as many of the largest container as possible, then the next largest and so on. Pantumsinchai and Knowles provided a general algorithm for solving the problem. The algorithm was designed for an ordering cost of zero and a modification was suggested for positive ordering cost. Since under significant discounts, the restricted policy yields optimal or near-optimal solutions, Pantumsinchai and Knowles devised an efficient algorithm for it.

Khouja [26] considered an SPP in which an emergency supply option exists. In the case of a shortage, a proportion of customers are willing to wait for emergency supply. Unit cost from the emergency supply is $R$, where $C < R < P+S$. Khouja maximized $E(\pi)$ and $P_B$. For maximizing $E(\pi)$, Khouja derived the sufficient optimality condition for any demand distribution. Let $t$ be proportion of customers who are willing to wait, for maximizing $P_B$, Khouja identified two cases: (a) $t(P-R)>(1-t)S$ for which $Q^*$ is independent of the demand distribution and is provided in closed form and (b) $t(P-R) < (1-t)S$ for which $Q^*$ is dependent on the demand distribution. Khouja derived closed-form expressions for $Q^*$ for exponential and uniform demand distributions.

Lin and Kroll [27] considered all-units and incremental quantity discounts and dual performance measures. The dual performance measures resulted in the objective function "maximize the expected profit subject to a constraint that the probability of achieving a target profit level is no less than a predetermined risk level", which was proposed by Ismail and Louderback [6]. Lin and Kroll treated two cases of shortage costs: $S = 0$ and $S>0$ which with the two quantity discounts resulted in four models: (a) all-unit discount and $S = 0$, (b) incremental discount and $S = 0$, (c) all-unit discount and $S>0$ and (d) incremental discount and $S>0$. Lin and Kroll provided simple algorithms for solving the first two cases and suggested using techniques such as a Lagrangian multiplier or penalty functions for the last two cases.

Kabak and Weinberg [25] proposed three extensions to the SPP: (1) supply of inventory is a random variable due to a supplier with variable capabilities, (2) suppliers are charged a penalty for not being able to meet contract obligations; the penalty can be fixed or proportional to the quantity of shortage and (3) a secondary supplier can supply additional units when the primary supplier can't provide $Q^*$. The secondary supplier charges a higher unit price.

### 3.3. Extensions to different news-vendor pricing policies and discounting structures

Researchers have suggested SPP extensions in which demand is price dependent [29–34]. Whitin [34] assumed that the expected amount demanded is a function of price and using incremental analysis, derived the necessary optimality condition. He then provided closed-form expressions for the optimal price ($P^*$), which is used to find $Q^*$ for a demand with a rectangular distribution. Mills [32] also assumed demand is a random variable with an expected value that is decreasing in price and with constant variance. Mills derived the necessary optimality conditions and showed that, under reasonable assumptions, $P^*$ under uncertainty is less than the riskless $P^*$. He also provided further analysis for the case of demand with rectangular distribution.

Lau and Lau [31] introduced a model in which the NV has the option of decreasing $P$ in order to increase demand. Lau and Lau analyzed two cases for demand:

1. Case A; the demand is given by a simple homoscedastic regression model $x=a-bP+\varepsilon$, where $a$ and $b$ are constants and $\varepsilon$ is normally distributed. The above equation implies a normally distributed demand which decreases linearly with $P$.
2. Case B; the demand distribution is constructed using a combination of statistical data analysis and experts' subjective estimates. The 'method of moments' was used to fit the four-parameter Beta distribution to estimate demand.

For case A, Lau and Lau showed that $E(\pi)$ is unimodal and thus the golden section method can be used. For case B, there is no guarantee $E(\pi)$ is unimodal. Thus, Lau and Lau developed a search procedure for identifying local maximums. Lau and Lau also maximized $P_B$ and considered the two cases of $S = 0$ and $S>0$. For $S = 0$ and case A, Lau and Lau derived closed-form solutions for $Q^*$ and $P^*$. For $S = 0$ and case B, Lau and Lau developed a procedure for computing $P_B$ and used a search procedure for $Q^*$. For $S > 0$ and cases A and B, $P_B$ may not be unimodal. Lau and Lau developed procedures for computing $P_B$ and finding $Q^*$.

Polatoglu [33] also considered the simultaneous pricing and procurement decisions. Polatoglu identified few special cases of the demand process addressed in the literature: (i) an additive model in which the demand at price $P$ is $x(P) = \mu(P) + \varepsilon$, where $\mu(P)$ is the mean demand as a function of price and $\varepsilon$ is a random variable with a known distribution and $E[\varepsilon] = 0$, (ii) a multiplicative model in which $x(P) = \mu(P) \cdot \varepsilon$ where $E[\varepsilon] = 1$ and (iii) a riskless model in which $X(P) = \mu(P)$. Polatoglu analyzed the SPP under general demand uncertainty to reveal the fundamental properties of the model independent of the demand pattern. Polatoglu assumed an initial inventory of $I$, $\mu(P)$ is a monotone decreasing function of $P$ on $(0, \infty)$ and a fixed ordering cost of $k$. For linear expected demand ($\mu(P) = a - b \cdot P$, where $a$, $b > 0$) Polatoglu proved the unimodality of $E(\pi)$ for uniformly distributed additive demand and exponentially distributed multiplicative demand.

Khouja [30] solved an SPP in which multiple discounts are used to sell excess inventory. In this model, retailers progressively increase the discount until all excess inventory is sold. The discount prices are $P_i$, $i = 0, 1, \cdots, n$, where $P_i > P_{i+1}$. The quantity demanded at each discount price $P_i$ is a multiple $t_i$ of the quantity demanded at the regular price $P_0$. Khouja assumed that $t_i$, $i = 1, \cdots, n$ are known parameters and solved the problem under two objectives: (a) maximizing $E(\pi)$ and (b) maximizing $P_B$. Khouja showed that $E(\pi)$ is concave and derived the sufficient optimality condition for $Q$. For maximizing the probability of achieving a target profit of $B$, $P_B$, Khouja provided a closed-form expression for $Q^*$. Khouja [29] solved the multi-discount SPP when the supplier offers an all-units quantity discount. Khouja developed an algorithm for identifying $Q^*$ under the objective of maximizing $E(\pi)$. The algorithm may require several evaluations of numerical integrals.

### 3.4. Extensions to random yields

Scholars have suggested SPP extensions in which $Q$ contains defective units [35–41] or the available production capacity is a random variable [42,43]. Karlin [38] assumed that the number of good units in a lot is a random variable with a known probability distribution. Karlin limited the ordering decisions to two alternatives: (a) do not order and (b) order from a choice of set levels, which does not allow for a range of order sizes. Shih [41] assumed defective units are unsaleable and are returned to the manufacturer at his/her expense. He also assumed that the percentage of defectives ($\rho$) is a random variable with known probability distribution. Shih derived the expected cost function and provided proof of its convexity and derived the necessary optimality condition for $Q$ for any distribution of $x$ and $\rho$. Noori and Keller [39] obtained analytical results for $Q^*$ for uniformly and exponentially distributed demand. Ehrhardt and Taube [35] generalized Shih's model by dealing with general forms of holding and shortage cost instead of the linear case and derived the necessary optimality conditions. For uniformly distributed demand, Ehrhardt and Taube provided a closed-form expression for $Q^*$. Ehrhardt and Taube provided a heuristic when $\rho$ has a Beta distribution and demand follows a negative binomial distribution.

Gerchak et al. [36] also dealt with random yield and assumed that there is some existing initial stock, $I$. Also, Gerchak et al. allowed the cost to be proportional to $Q$ or to the net yield. Gerchak et al. showed that $E(\pi)$ is concave in $I$ and $Q$ and that there is a critical level of $I$ above which no order will be placed under certain yield, and this level is the same under random yield. Gerchak et al. also showed that unlike certain yield, the optimal policy under random yield is not of the 'order-up-to' type.

Henig and Gerchak [37] did not assume that the yield constitutes a proper fraction of $Q$ which makes the model applicable to situations where the input level and yield size are not measured in the same units. In one part of the analysis, Henig and Gerchak did not make any assumptions about the manner in which the yield depends on $Q$. In another part, Henig and Gerchak assumed stochastically proportional yield. Henig and Gerchak assumed that production costs depend on the realized yield and then generalized their results to where the production costs depend on both $Q$ and the realized yield. Henig and Gerchak assumed a beginning inventory of $I$ and analyzed the critical ordering point $I$ under certain and random yield. Henig and Gerchak showed that for a given $I$, if it is not optimal to order when yield is certain, it is also not optimal to order when yield is random.

Parlar and Wang [40] analyzed an SPP in which the NV uses two suppliers, each having random yield. Parlar and Wang assumed that the suppliers have different yield distributions and prices and used the stochastically proportional yield assumption. Diversification may still be useful since it may reduce the overall yield variability. Parlar and Wang proved the concavity of $E(\pi)$ for a general demand distribution and proposed an approximate solution technique for finding $Q_i^*$, $i = 1, 2$.

Ciarallo et al. [42] analyzed an SPP in which the uncertainty is a result of random capacity rather than yield. Ciarallo et al. assumed that because down time is uncertain, productive capacity is a random variable with a known distribution. Ciarallo et al. proved that under linear shortage and holding cost, the expected cost function is non-convex but unimodal which allows classical convex procedures to be used for optimization. Ciarallo et al. showed that the optimal policy in this case is identical to the classic SPP model. Jain and Silver [43] also assumed demand and supplier capacity to be random variables with a known probability distributions. Jain and Silver assumed that the NV can assure the availability of a given level of capacity by paying the supplier a premium ahead of time. The NV does not have the obligation of ordering the quantity that corresponds to the full utilization of dedicated capacity. Jain and Silver assumed that the cost of reserving capacity is a monotonically increasing convex function. The NV must decide on the level of dedicated capacity and on $Q$. Jain and Silver derived $E(\pi)$ which may have multiple local maxima as a function of dedicated capacity. For normally distributed demand and capacity, Jain and Silver developed a solution algorithm.

### 3.5. Extensions to different states of information about demand

Several authors analyzed SPPs in which the demand does not satisfy the classical assumptions of having a specific distribution with known parameters [2,44–50]. Other researchers studied the effects of increased variability of demand [51] or solved the SPP under simpler demand distributions [52]. Scarf [49] assumed that only the mean $\mu$ and variance $\sigma^2$ of demand are known and derived in closed-form the $Q^*$ which maximizes $E(\pi)$ against the worst possible distribution of demand.

Gallego and Moon [2] provided a simpler proof of Scarf's rule and derived, in closed-form, a simple lower bound on $E(\pi)$ with respect to all possible demand distributions. Gallego and Moon also provided four extensions to the distribution free SPP:

1. A recourse case in which there is a second purchasing opportunity after observing demand. After ordering $Q$ and finding that $x > Q$, an additional order for $x - Q$ is placed at a higher cost. Gallego and Moon derived closed-form expressions for $Q^*$.
2. A fixed cost case in which a fixed cost is charged for placing an order and there is an initial inventory. Gallego and Moon derived a simple ordering rule for $Q^*$.
3. A random yield case in which the number of good units out of $Q$ is a random variable $G(Q)$. Gallego and Moon assumed that each unit has a $\rho$ probability of being good, which implies that $G(Q)$ is a binomial random variable. Gallego and Moon derived a closed-form expression for $Q^*$ and a lower bound on $E(\pi^*)$.
4. A multi-product case in which products compete for a scarce resource. Gallego and Moon used a budget constraint. This problem is sometimes referred to as the stochastic product mix problem problem [53]. Gallego and Moon formed the Lagrangian function and developed a solution algorithm.

Moon and Choi [46] maximized $E(\pi)$ against the worst possible distribution of demand for a distribution free SPP with balking. Moon and Choi assumed that once the inventory level falls to a level $k$ or less, the probability that an arriving customer will buy the product drops from 1 to $L$. Moon and Choi developed the necessary and sufficient optimality condition which requires a line search for $Q^*$. Moon and Choi also solved the problem in the presence of an initial inventory and ordering cost and developed the necessary and sufficient condition for the optimal inventory level ($S^*$).

Reyniers [48] developed a high-low search algorithm for the SPP under uncertainty rather than risk. Actual demand is a constant $D$ which is unknown but has a known lower bound $D_L$ and upper bound $D_U$. Ordering a quantity of $Q$ can be interpreted as making a guess about D. The problem becomes finding a sequence of guesses such that in the most adverse demand conditions, $E(\pi)$ is maximized. Reyniers dealt with asymmetric information feedback where demand is found only when supply exceeds demand. Otherwise, only a lower bound on demand is observed. Reyniers devised an algorithm for finding $D$ at maximum profit using the theory of high-low search. In the original high-low search games, a keeper hides an integer and the payoff is the number of guesses required by the seeker to find this integer given that the seeker is told after each guess if it was too high, too low, or exact.

Petrovic et al. [47] developed two fuzzy models to deal with uncertainty in the SPP. The models address two cases: (a) imprecisely described discrete demand but precise $C_o$ and $C_u$ and (b) imprecise demand and imprecisely estimated $C_o$ and $C_u$. In both models demand is based on subjective judgment and can be vaguely expressed by statement such as "demand is much larger than $d_l$". Petrovic et al. suggested that fuzzy set theory provides the appropriate framework to describe and treat uncertainty related to imprecision of natural language expressions. For both cases $Q^*$, is determined by one dimensional search. Petrovic et al. provided numerical examples for both cases. While imprecise demand alone did not lead to a large change in $Q^*$ from the classical case, the fuzziness in the overage and underage costs did. This conclusion may be only valid for the numerical examples.

Gerchak and Mossman [44] analyzed the effects of demand randomness on $Q^*$ and $E(\pi)$. While the literature has some ordinal results concerning the direction of change because of randomness, the authors, provide cardinal results about the changes in $Q^*$. Thus, the authors provided statements about the magnitude of change and not only direction. To model increases in randomness, the authors used a mean $\mu$ preserving transformation

$$x_\alpha = \alpha x + (1-\alpha)\mu, \quad \alpha \geq 0 \tag{6}$$

which is often used in microeconomics. The authors showed that the optimal order quantity (now denoted $Q^*_\alpha$) inherits the exact transformation of the random demand

$$Q*_\alpha = \alpha Q^*_\alpha + (1-\alpha)\mu, \quad \text{for all } \alpha \geq 0, \tag{7}$$

which implies that the optimal order quantity is increasing in demand variability if that quantity is below the mean, which is rather plausible if the demand distribution is not too positively skewed. The authors showed that risk pooling, i.e. aggregating several random demands into one, may not result in a reduction in $Q^*$ or at least move it closer to the mean demand as intuition suggests.

Shih [50] and Hill [45] applied a Bayesian methodology to the SPP. The Bayesian approach deals with a stochastic decision-making environment in which random variables follow known distributions with unknown but fixed parameters. Using collateral data and/or subjective assessment, a 'prior' distribution for the unknown parameter is constructed. As new data becomes available, the prior distribution is updated and a 'posterior' distribution of the unknown parameter is generated. This posterior distribution is then used to obtain a new $Q$. Shih [50] used a Gamma prior distribution of the unknown mean and exponentially distributed demand. Hill [45] used a uniform prior over some permitted parameter range and exponential, Poisson and binomial demand distributions. Hill showed that the application of the Bayesian methodology, produces better results than using a single point estimate of the unknown parameter under exponential distribution. Hill also showed that the same conclusion holds over a wide range of parameter values for the Poisson and binomial.

Ridder et al. [51] studied the effects of demand variability on $E(\pi^*)$. Intuition suggests that higher demand variability results in larger variances and smaller $E(\pi^*)$. Song [54] proved this assertion for a class of problems which included the SPP for many commonly used demand distributions including the normal. Ridder et al. showed that Song's conclusion is not always true and used stochastic dominance to characterize the conditions under which the opposite relationship between demand variability and $E(\pi^*)$ is true. Ridder et al. developed the sufficient conditions under which higher demand variability will lead to an increase in $E(\pi^*)$.

Kumaran and Achary [52] solved the SPP for a demand with a generalized λ-type distribution (GLD). Kumaran and Achary pointed out that the main problem with some other distributions is that they require closed-form expressions for the cumulative distribution function (cdf), inverse cdf and the loss function. GLD is a four-parameter family of distributions which, among its advantages, are its ability to assume different shapes both symmetric and skewed and has simple closed-form expressions for the inverse cdf and the loss function. Using demand with GLD, Kumaran and Achary provided closed-form expressions for $Q^*$ and $1-F(Q^*)$. Based on 80 test cases, Kumaran and Achary showed that $Q^*$ and $1-F(Q^*)$ obtained under the GLD approximation to normal, exponential and Gamma distributed demands are quite accurate.

### 3.6. Extensions to constrained multi-products

Several authors developed constrained multi-product extensions to the SPP [55–59]. Silver et al. [4] provided detailed mathematical analysis of the constrained multi-product SPP and a review of many extensions related to it. Some of these extensions may fall outside the domain of this review. We focus our review on those models that are most related to the SPP and are not fully covered by Silver et al. Hadley and Whitin [55] solved the multi-product SPP under a storage (or budget) constraint. Hadley and Whitin developed two algorithms. The first is based on a search for the Lagrangian multiplier that satisfies the necessary conditions. Results of this algorithm will have to be rounded to integers and thus are suitable when the $Q_i^*$'s are large. For the case when the $Q_i^*$'s are small and rounding may have a significant impact on $E(\pi^*)$, Hadley and Whitin developed a marginal analysis approach to find an integer solution. Nahmias and Schmidt [59] provided four heuristics for solving the single-constraint SPP under normally distributed demand. One of the heuristics required fewer computations and provided good solutions relative to the Lagrange method. The procedure is useful for continuous $Q$'s and is thus appropriate for moderate-to high demand items. For a 5000-item problem, Nahmias and Schmidt's heuristic required a computing time of 5 s versus 2 h for the Lagrange method on a DEC 2060 system.

Lau and Lau [58] solved a multi-product multi-constraint SPP. Since evaluating $E(\pi)$ involves many integrals which is time consuming, a direct search procedure to numerically evaluate $E(\pi)$ is inappropriate. Also, since a typical newsstand will have a large number of products and much smaller number of constraints, the $N$-variable 'primal problem' should be converted to $M$-variable 'dual problem'. Lau and Lau developed a procedure based on the 'active set methods', tested the procedure against state-of-the-art nonlinear programming software and found that their procedure was much faster and provided better quality solutions.

Lau and Lau [57] showed that Hadley and Whitin's procedure for the single-constraint SPP based on the Lagrangian multiplier works only for a limited class of demand distributions. Lau and Lau observed that if the demand distribution for product $i$ has a lower bound greater than zero, then $F_i^{-1}(0)$ is indeterminate and the solution procedure breaks down. Lau and Lau developed a procedure for dealing with this case.

Khouja and Mehrez [56] extended the model proposed by Khouja [30] to the multi-product case. Thus, this model deals with an NV selling many products, offering progressively steeper discounts and operating under a budget constraint. Using the concavity properties of the single-item case proved by Khouja [30], Khouja and Mehrez modified Hadley and Whitin's procedures to deal with this case. The first procedure deals with large $Q_i^*$'s. The second procedure is a simple modification of the marginal analysis approach procedure used when the $Q_i^*$'s are small and rounding may have a significant impact on $E(\pi^*)$.

### 3.7. Extensions to multi-product models with substitution

Several authors proposed extensions to the SPP in which customers substitute another product from the same NV for the product they demand in case of a shortage [60–64] or substitute a product from the competition [65,66]. Another extension dealt with economic substitution [67].

Pentico [64] addressed what is known as the assortment problem in which there are a set of sizes $N=\{1, 2, \cdots, n\}$ of a product. Size 1 is the largest and $n$ is the smallest. The NV will stock only a subset of sizes because of a storage constraint. Demand for an unstocked size $i$ will be satisfied from a larger stocked size $j$ with a substitution cost $b_{ij}$. The problem is to find the set of sizes to stock and the order quantities that will minimize the total cost. Pentico assumed that demands are independent random variables with known distributions and that demand for any size will be supplied from the smallest possible stocked size. Pentico formulated and solved the problem using dynamic programming. Silver et al. [4] provide further references for the assortment problem.

Parlar and Goyal [63] developed a two-product SPP in which each product can substitute for the other in case of a shortage. Parlar and Goyal assumed that the salvage value and the lost sales penalties are zero and that substitution occurs according to fixed probabilities. Parlar and Goyal derived the expression for $E(\pi)$ and showed that under a certain condition, it is strictly concave. Parlar and Goyal derived the necessary optimality conditions and using some qualitative analysis provided good starting values to obtain $Q_i^*$, $i = 1, 2$ using an iterative solution technique such as the Method of Newton. Khouja et al. [62] revisited the two-item SPP with substitutability. Khouja et al. derived $E(\pi)$ for positive salvage values and penalty shortage costs but were unable to prove concavity. Khouja et al. developed upper and lower bounds on the $Q_i^*$'s and developed a Monte Carlo simulation to identify them.

Gerchak et al. [61] extended the SPP to a case of a product with two grades with downward demand substitution and production process having random yields. Gerchak et al. analyzed two models. In model I, a single production process yields a random quantity of usable products of which the quantity of the higher grade product constitutes a random function. In model II, two production processes are used. The first process is similar to the process in model I. The second process produces only lower grade products but their yield is random. Gerchak et al. analyzed both models under known demand. For model I, Gerchak et al. showed that $E(\pi)$ is concave in $Q$ and derived the sufficient optimality condition. For model II, Gerchak et al. showed that $E(\pi)$ is concave in $Q_1$ (produced on the first process) and $Q_2$ (produced on the second process) and derived the sufficient optimality conditions.

Bassok et al. [60] developed a multi-product SPP model with substitution. Bassok et al. assumed $N$ products and $N$ demand classes with full downward substitution. Bassok et al. assumed that the substitution cost is proportional to the quantity substituted. The proposed problem is a two stage decision model. In stage I, the NV decides on the $Q_i^*$'s. In stage II, after observing demand, the NV decides how to allocate the $Q_i^*$'s among the $N$ demand classes. Bassok et al. developed a greedy algorithm to find the optimal allocation and showed that $E(\pi)$ is concave and submodular which enabled them to prove several properties of the optimal policy. Bassok et al. developed expressions for the first differentials of $E(\pi)$ which are useful in developing any gradient based algorithm. Using a 2-product problem, Bassok et al. demonstrated that significant gains can be obtained by considering substitution.

Parlar [66] analyzed a two-NV SPP in which when one has a shortage, a fraction of his/her customers switches to the other NV. Thus, each NV's $Q$ affects the other NV's $E(\pi)$ and the decision of each NV can't be treated in isolation of the other. Parlar used game theoretic ideas to analyze the decision-making strategies of the NVs. The NVs were assumed to have knowledge of the demand densities, substitution rates, and other parameter values. Parlar analyzed the decisions under three assumptions about the behavior of the NVs:

1. No cooperation between the NVs. If the NVs are

rational (no NV will risk lowering his/her $E(\pi)$ for the purpose of damaging the competitor), then the NVs may adopt a Nash strategy.
2. One irrational NV. One NV attempts to inflict the maximum damage on the other. Parlar showed that the maximin strategy of the other NV reduces to strategy in the classical SPP.
3. Perfect cooperation. In this case, one NV does not incur a penalty if the other NV satisfies the former's demand.

Lippman and McCardle [65] investigated the effects of competition on inventory levels in the SPP. Their goal was to determine whether the $Q_i^*$'s in the multi-firm industry can still be characterized as fractiles of the demand distributions as is the case in the classical SPP. In addition, Lippman and McCardle compared the equilibrium inventory levels and stockout probabilities of the multi-firm competitive solution and the classical monopolist's solution. Lippman and McCardle assumed that demand is split among several firms. To isolate the pure impact of competition, Lippman and McCardle assumed that the aggregate demand does not change with the total number of firms. Each firm's strategy is characterized by its $Q$. Lippman and McCardle assumed no price competition because each firm charges a preset price. The existence of competition means that a customer who finds the shelves empty at one firm may visit another to satisfy his/her demand. Thus, there are two aspects of demand: the initial allocation and the reallocation. Initial allocation does not depend on the order quantities but rather a rule known to all. The reallocation refers to the portion of excess demand that is reallocated to other firms. Thus, firm A may increase its $Q$ not only to capture the excess demand from firm B but also to restrain firm B from ordering too much. Lippman and McCardle introduced four splitting rules for the initial allocation of demand: deterministic splitting, simple random splitting, incremental random splitting, and independent random demands. For an industry made up of two firms, Lippman and McCardle showed that if all excess demand is reallocated, competition never leads to a decrease in industry inventory. Also, under deterministic splitting with each firm's share increasing in the total industry demand, competition does not alter industry inventory, there is a unique equilibrium and the inventory ordering rule is represented as a fractile of the effective demand distribution, which extends the results of the classical SPP. For larger than two-firm industry, Lippman and McCardle showed that if demand is split according to a randomized rule in which all firms are treated identically, then competition drives expected industry profit to zero as the number of firms increases.

Deuermeyer [67] introduced the concept of economic substitution in the SPP which means that $Q_i$ ($i = 1, 2, \cdots, n$) is a non-increasing function of the on-hand inventories of the other products. In this case, some amount of product $i$ can be replaced by an increase in the amounts of other products kept on hand in terms of the economic benefits realized by the NV. Thus, economic substitution in this model does not imply that product $j$ can be used to satisfy demand for product $i$. Deuermeyer showed that the optimal policy uses the economic substitution property and that the optimal inventory level (on-hand + $Q_i^*$) is more responsive to changes in newer inventory than older inventory.

### 3.8. Extensions to multi-echelon systems

Several multi-echelon extensions to the SPP have been suggested [68–71,53,72–75]. Gerchak and Henig [70] formulated a single period model for selecting optimal component stock levels in an assemble-to-order system. For a given component stock level, Gerchak and Henig determined the revenue maximization allocation of common components between products. This information is in turn used to select the optimal stock levels.

Jonsson and Silver [73] also dealt with assemble-to-order environment where some of the components are unique to specific end items while others are common to two or more end items. Jonsson and Silver assumed that components must be ordered before demands for end items become known. Jonsson and Silver also assumed that demands for end items are normally distributed and their values become known before final assembly operations begin. Jonsson and Silver addressed the problem of deciding on component quantities under a budget constraint so as to maximize the expected number of units of end items sold. The problem was addressed for two end items, each composed of two components, and one of the components is common to both end items. Jonsson and Silver developed a simple heuristic for solving the problem. Jonsson and Silver [74] extended their earlier model to deal with many components and end items. Jonsson and Silver formulated the problem as a two-stage stochastic programming problem with a recourse which turns out to be extremely difficult to solve optimally. Jonsson and Silver developed three heuristics for solving the problem under some simplifying assumptions. One of the heuristics performed well for the practical case of continuous demand distributions for end items and large budgets. Jonsson et al. [72] used scenario aggregation technique for solving the two-stage stochastic programming problem formulated by Jonsson and Silver. The basic idea behind scenario aggregation technique is to consider only a relatively small subset of the typically large number of stochastic demand outcomes. For each subset, the optimal values of the decision variables can be easily found. Scenario aggregation ensures that an implementable solution is then obtained using an iterative scheme. Jonsson et al. found that in 50% of the problems studied, scenario aggregation identified the optimal solution.

Gerchak and Zhang [71] developed a two-echelon SPP in which decisions have to be made on whether to hold inventories in the form of raw materials or finished products. Holding raw material is less costly but if demand turns out to be high then a fraction of demand is lost since some customers might not be willing to wait for the conversion of raw materials into finished goods. Gerchak and Zhang assumed that there is an initial inventory at both stages and derived $E(\pi)$ as a function of the inventory levels and proved its concavity. Gerchak and Zhang then derived the optimal inventory policy. Their conclusion can be summarized as: if both initial stocks are high nothing will be ordered, and, depending on the proportion of finished products in the total stock, some raw material might be processed. If the stock of finished products is high but the combined stock is low, only raw materials should ordered. If both stocks are low, both items should ordered up to constant level derived by Gerchak and Zhang.

Eynan and Rosenblatt [69] further generalized Gerchak and Zhang's model. Eynan and Rosenblatt used the SPP to evaluate an assembly in advance (AIA) versus an assemble to order (ATO) strategies in a two-echelon production system. Productions cost for AIA items is lower than ATO items because there is no need to expedite and production can be executed well. The trade-off is between carrying more costly finished goods inventory with the risk of having some units undemanded or carrying components inventory and assembling it to order at a higher production cost. The decision variables are the amounts of inventory to hold at the raw material stage $Q_r$ and the finished goods stage $Q_s$. Eynan and Rosenblatt showed that $E(\pi)$ is concave and derived the necessary optimality conditions for $Q_r$ and $Q_s$. Eynan and Rosenblatt provided conditions under which each strategy is optimal. They then solved the problem when the assembly time is not negligible resulting in some lost sales when finished goods inventory is exhausted because some customers will not wait. In addition, they solved the problem under three types of budgetary constraints. Moon and Choi [75] extended Eynan and Rosenblatt's model to the distribution free case where only the mean and standard deviation of the demands are known. Moon and Choi assumed that there is no initial inventories and all customers will wait for the conversion process and derived closed form expressions for $Q_r^*$ and $Q_s^*$. Moon and Choi also treated the problem under a budget constraint using the Lagrange method. Other references to models related to this area of the SPP can be found in Silver et al. [4].

### *3.9. Extensions to multi-location models*

Multi-location SPP extensions can be divided into two types: (1) all locations have the same selling season [76–80] and (2) the selling seasons of the different locations lag each other [81].

Eppen [79] analyzed the effects of centralization on the multi-location SPP. In this model, there are $N$ retail centers which raises the opportunity for centralization. Eppen compared the expected cost of two configurations: (a) a decentralized system in which a separate inventory is kept at each center and (b) a centralized system in which inventory is kept at central warehouse. Eppen assumed normal demand distribution and linear holding and penalty costs and showed that the expected cost of the decentralized facilities exceeds that of the centralized facility with the difference depending on the correlation of demands. For uncorrelated and identically distributed demands, the expected cost of the centralized facility increases as the square root of the number of consolidated centers. Stulman [80] analyzed Eppen's model [79] with Poisson demand distributions and replaced the penalty shortage cost with a service level constraint on meeting demand at each location. Using a first come first served rule, Stulman found the optimal starting inventory for the centralized and decentralized facilities. Stulman showed that when each location's demand can be approximated by a normal distribution and under some conditions, the centralized system requires less starting inventory than the decentralized system and is less costly. Chen and Lin [77] showed that Eppen's conclusion holds for any probability distribution and concave holding and penalty cost functions. Chen and Lin [78] provided a counter example to the conclusion of Stulman [80]. Chen and Lin showed that under Stulman's assumptions, a centralized two-location system with a maximum acceptable probability of stockout of 86% at each location has a higher starting inventory than a decentralized system. Chang and Lin [76] noted that previous work did not take into account the transportation cost when a centralized facility is used and reworked the model with the addition of transportation cost.

Kouvelis and Gutierrez [81] observed that an NV can exploit the difference in timing of selling seasons of geographically dispersed markets. For example, a US garment maker can sell his/her remaining summer fashion in Australia where summer is about to begin. Kouvelis and Gutierrez considered a secondary market whose selling season follows the season of the primary market. Kouvelis and Gutierrez studied coordination among markets under exchange rate uncertainty. A centralized policy is one in which a corporate planner decides on how much to produce at each market and how much excess inventory to ship from the primary to the secondary market. A decentralized control policy is one in which each market is treated as a profit center and the only coordination mechanism is a constant transfer price ($C_T$) at which the secondary market can buy items from the primary one. Kouvelis and Gutierrez showed that a centralized policy is more profitable for any $C_T$ and that the difference in $E(\pi)$ can be significant. However, Kouvelis and Gutierrez observed that decentralization is preferable from a managerial point of view for accountability. Kouvelis and Gutierrez proposed a third policy in which decisions are still decentralized but the transfer between markets is handled by a third corporate unit. The unit buys goods from the primary market using a nonlinear pricing scheme developed by Kouvelis and Gutierrez and sells them to the secondary market at the salvage value of the primary market. Kouvelis and Gutierrez showed that this policy is equally profitably as the centralized policy.

### 3.10. Extensions to models with more than one period to prepare for the selling season

The idea behind these models is that there may be many periods to produce or purchase the items which will be sold in a single season. The question becomes how should the NV plan production for items produced in-house or what orders should be placed with the suppliers as the selling season draws closer. Silver et al. [4] provided a good review and an extensive list of references for this extension of the SPP. To avoid duplication, we focus our attention on a subset of representative models [82–86]. Murray and Silver [86] assumed there are $m$ opportunities to purchase the item at pre-specified points in time ($T_1$, $T_2$, $\cdots$, $T_m$) and that the cost depends on the purchase time. The item is displayed in competition with others of its type sold by other NVs. Murray and Silver assumed that the total number of customers demanding an item of this type between $T_i$ and $T_{i+1}$, is known. However, the chance that a customer will buy the NV's product, denoted $p$, is unknown and is fixed for the season. As the season progresses the NV's knowledge about the value of $p$ improves. Thus, the sales potential of the product is treated as a subjective random variable whose distribution is changed adaptively using Bayes's rule as the sales history unfolds. Murray and Silver used the two-parameter Beta distribution to represent the prior distribution. The problem becomes finding $Q_i$'s at $T_1$, $T_2$, $\cdots$, $T_m$ which will maximize $E(\pi)$. Murray and Silver formulated the problem as a dynamic program and provided computational shortcuts for certain types of the problem.

Hausman and Peterson [83] extended Murray and Silver's model to the case of multiple products and limited production capacity in each period. Hausman and Peterson assumed that ratios of successive forecasts of total orders for a seasonal product are mutually independent Lognormal variates whose parameters can be estimated by analysis of historical forecast data. Hausman and Peterson formulated both the single and multi-product cases as dynamic programming problems and developed and compared three heuristics to solve the multi-product production planning problem.

Bitran et al. [82] dealt with a system that produces several families of style goods. A family is defined as a set of items consuming the same amount of resources and sharing the same setup. Bitran et al. assumed that the setup cost associated with changeover from one family to the next is large enough that managers attempt to produce each family once in the planning horizon. Also, Bitran et al. assumed that the mean demand for each family is invariant over the horizon whereas item demands are forecasted in each period. Demand occurs in the last season of the horizon and demand estimates for items are revised every period. The problem is finding item production quantities which will maximize $E(\pi)$. Bitran et al. assumed the demand of items in a family follow a joint normal distribution and that each period has limited production capacity. The problem was formulated as a difficult stochastic mixed integer programming problem and by exploiting its hierarchical structure (families and then items), Bitran et al. formulated and solved a deterministic mixed integer programming problem which provided an approximate solution.

Matsuo [85] observed that a limitation of Bitran et al.'s model is that it included discrete production periods and each family is assigned to exactly one period which works well only if the number of families is much larger than the number of periods. Also, the complexity of Bitran et al.'s method made sensitivity analysis difficult. To avoid the limitations of Bitran et al.'s model, Matsuo used a continuous treatment of time and formulated the problem as a two-stage stochastic sequencing model. In stage I, a sequence of production quantities of families is determined at the beginning of planning horizon. In stage II, the production quantities of items in each family are determined using the revised demand forecast. Matsuo developed and tested a heuristic procedure for solving the problem.

Kodama [84] analyzed an SPP in which partial returns up to a level $R$ are allowed in case of a surplus and additional purchases up to a level $A$ are allowed in case of a shortage. Kodama assumed that surplus inventory can carry over to the next period and derived the optimality condition. Kodama noted that the

SPP is a special case of the model when surplus inventory can't be carried over.

### 3.11. Other extensions

Other extensions have been proposed to the SPP. Pfeifer [87] introduced an extension to yield management. Airlines often offer discount-fare tickets which must be purchased before a certain time from flight departure and the rest of the tickets are sold at full-fare prices. Yield management is the process by which the discount fares are allocated to scheduled flights for the purposes of increasing revenues. Let $R_D$ be the discount fare, $R_F$ be the full fare, $q$ be the number of seats to be made available at $R_D$ and $Q$ be the total number of seats available. The problem is to find $q$ which maximizes $E(\pi)$. Pfeifer analyzed two strategies: (a) a permanent allocation in which $q$ is determined once and remains unchanged and (b) a continuous review in which the question changes from what should $q$ be to when to stop sales at $R_D$ and begin at $R_F$. Let $P1$ be the probability that the $(q+1)$st potential customer will purchase a seat at $R_D$ but will be lost if $R_D$ is no longer available and $P2$ be the probability that the remaining $Q-q-1$ seats will satisfy all subsequent customers willing to pay $R_F$. Obviously $P1$ and $P2$ are a function of $q$ and will have to be estimated. Pfeifer used marginal analysis to provide the optimality condition for $q$. Pfeifer also showed that the SPP can be used to solve the problem. The key is not to think in terms of $q$ but rather in terms of $Q-q$, the number of seats to reserve for sale at $R_F$. With $Q-q$ seats held, the question is how many will not be sold which is the classical SPP. The demand distribution becomes the conditional distribution of subsequent full-fare demand, given that $q$ discount-fare seats have been sold.

Weatherford and Pfeifer [3] used the SPP to evaluate the economic value of advanced booking of orders (ABO) which is the practice of selling some units of a good or service in advance of their actual availability. Discount airline tickets and season tickets in advance of athletic games are examples of ABO. ABO have two benefits: (a) providing early information about demand and (b) capturing customers who will buy only if the discount is offered. For normally distributed demand, Weatherford and Pfeifer quantified the benefits of ABO which include: (a) benefits from improved yield management, (b) benefits from improved forecasting which leads to improved $Q^*$ and (c) benefits that accrue because management has the option of stopping the venture if the ABO shows disappointing results.

Lau [88] observed that even though there is a simple rule to compute $Q^*$ in the SPP, there are no formulas for computing $E(\pi)$. He derived simple formulas for computing $E(\pi)$ for uniformly and exponentially distributed demands and identified a near closed-form equation which was originally considered by Hadley and Whitin [55] for the normal distribution.

Ward et al. [89] pointed out the impracticality of theoretical approaches to the SPP. The applied approach may compromise theoretical completeness and assumptions' validity in favor of transparency and simplicity. Ward et al. pointed out that specifying the demand distribution is difficult and suggested working with an approximate discrete distribution. Also, Ward et al. pointed out that because model parameters are usually rough estimates, sensitivity analysis is needed. Ward et al. illustrated an applied approach to the SPP based on partial enumeration of $E(\pi)$ under a discrete demand distribution. The use of the discrete distribution is advocated because it is easier to specify and it can be subjectively adjusted to account for the cases when demand exceeded $Q$. Marginal analysis was used to determine $Q^*$.

Gerchak and Wang [90] solved an SPP in which demand depends randomly on the starting inventory level ($I$) which is the sum of the initial inventory ($I_0$) and the order quantity $Q$. Gerchak and Wang assumed that demand is given by $x=H(I)\cdot W$, where $H(I)$ is an increasing concave function of $I$, and $W$ is a nonnegative random variable with a known probability distribution. Gerchak and Wang proved that $E(\pi)$ is concave in $Q$ and derived the optimal policy in terms of whether to order in terms of $I_0$ and how much.

Some related models to the SPP are those in which the goal is to find when to discount in the SPP. Recently, Feng and Gallego [91] developed a model in which the capacity decisions for the period are fixed. Thus, revenue maximization becomes the objective. Feng and Gallego assumed that demand is price-sensitive and the NV knows the expected demand rate at certain prescribed prices. Feng and Gallego's goal was to determine the optimal time and direction of the price change.

## 4. Discussion

While many extensions of the SPP have been proposed, very little comparative work has been done. As described in Section 3.1, the SPP has been solved under many objectives. However, little work has been performed on comparing the results obtained under these differing objectives. For example, under what conditions in terms of demand distribution and problem parameters do the objectives of maximizing $E(\pi)$ and $P_B$ lead to similar or different $Q^*$. Additionally, while many objectives have been suggested, little attention has been given to their validity and suitability under different environments.

Extensions to different supplier and NV pricing policies are similar to extensions of the classical economic order quantity (EOQ) model proposed in the literature. The models extending the SPP to price-dependent demand make some assumptions which may be problematic from a practical point of view. The assumption that the NV will select a single price and quantity which maximizes expected profit may lead to an unrealistic problem formulation. The assumption is that the NV offers the product for a price of $P^*$ and if not all of $Q^*$ is demanded at $P^*$, the NV discounts the remaining units to a salvage value $V$. This policy is inferior to one in which the NV uses more than one discount, say for example, prices of $P^*$, $0.8P^*$, $0.6P^*$ and then $V$ (assuming $0.6P^* > V$) are used. This type of pricing policy was proposed by Khouja [29,30] but the author ignored price–demand relationships and assumed that a given demand multiplier can be sold at each discount price. A more realistic formulation is one in which the NV uses multiple prices and the quantity demanded at each price is based on some price–demand relationship. Another factor that should be taken into account is the fixed cost of discounting. In general, firms may incur some fixed cost for discounting their products which is especially true for retailers. The fixed discounting cost for retailers results from the need to advertise the discount and mark down the items discounted.

Extensions dealing with random yields also mirror those proposed for the EOQ and other deterministic inventory models. Many of the SPP extensions in this area are similar. However, there are some important extensions which have been proposed for the EOQ but not for the SPP. Among the most important of these extensions are ones in which product quality deteriorates with increased order quantity, $Q$ [92]. Using the EOQ framework, Porteus [92] assumed the production process to be perfectly functioning at the start of production of a lot. With the production of each unit the process may shift toward 'out-of-control'. In other words, the process follows a two-state Markov chain during the production of a lot, with transition occurring with each unit produced. This assumption leads to the proportion of defectives being increasing in $Q$. The impact of this observation may significantly reduce $Q^*$ in the SPP as is the case with the EOQ model.

While the models dealing with the SPP under substitution and different states of information about demand are discussed in different sections, there is an important relationship between them. Substitutability may be used to manage products with high demand variability and large overage cost. The idea is that when demand can be shifted from a product with high demand variability and large overage cost to a product with small overage cost, the total expected profit may be increased. Also, the models on substitutability would benefit from an empirical evaluation of the true patterns of substitution used by consumers. Most models assume downward substitution; however, the validity of this assumption cannot be asserted. Some of the literature on substitutability in economics may be useful in identifying realistic substitution patterns. Another issue raised by substitution is the difficulty it poses in terms of estimating demand for different products that are substitutes. If the NV offers a number of products that are substitutes, then how can one differentiate between the demand for a certain product as being a true demand for it or demand for another product for which there is a shortage? The models provide no means for handling this difficulty created by substitutability.

Extensions to multi-echelon systems do not seem to fit well within the classical assumptions of the SPP model. Many of these models deal with an assembly type operations. The type of products that are assembled, except for very high tech products, are not the type of products which are thought of in dealing with the SPP. Actually, some of these extensions were solved as a first step approximation to multi-period models.

A number of papers on the SPP have been motivated by the desire to develop practical models [15,19,20,89]. Thakkar et al. [20] thought that return on investment (ROI) is a better criteria than $E(\pi)$ and $P_B$. Later, Magee [19] and then Anvari [15] suggested that the capital-asset pricing model (CAPM) is the appropriate framework for the SPP. Ward et al. [89] suggested that working with an approximate discrete distribution and performing sensitivity analysis are needed to have a practical model. Without some empirical work examining real life objectives of managers and the availability of information about demand, the practicality of these models cannot be assessed.

## 5. Conclusion and suggestions for future research

Interest in the SPP has increased over the past 40 years. This interest can be attributed in part to the increased dominance of service industries for which the SPP is very applicable in both retailing and pure service organizations such as air transportation. Also, the reduction in product life cycles brought about by technological advances makes the SPP more relevant.

Further areas of SPP research include a joint determination of the optimal order quantity and the discounting policy which involves deciding on the number of discounts to offer and their magnitude. Substitution is another area of research. While most models assume downward substitution, the multi-product SPP under general substitution has not yet been addressed.

Another extension is the incorporation of the effects of advertising in the SPP. While researchers assumed that the quantity demanded is a function of price, this assumption did not take into account the ability of the NV to influence the quantity demanded through advertising. Other extensions may combine two or more of the extensions discussed in this paper to further generalize the SPP. The complexity of the problem will increase and heuristics procedures may have to be used to find good solutions.

## Acknowledgements

The author would like to thank the referees for their helpful suggestions.

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