# Low-Variance Retrial Times in a Multiserver Queueing Model

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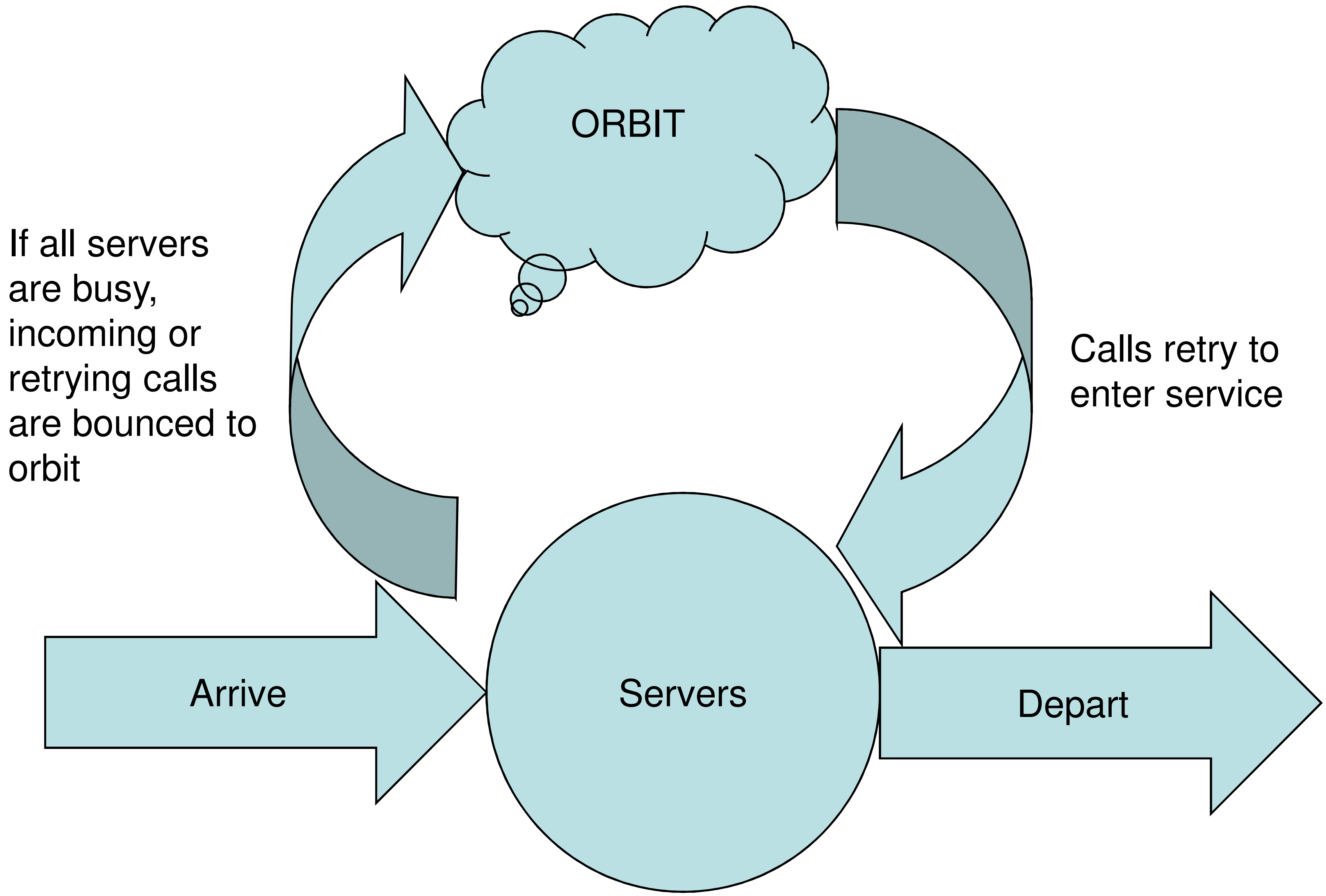
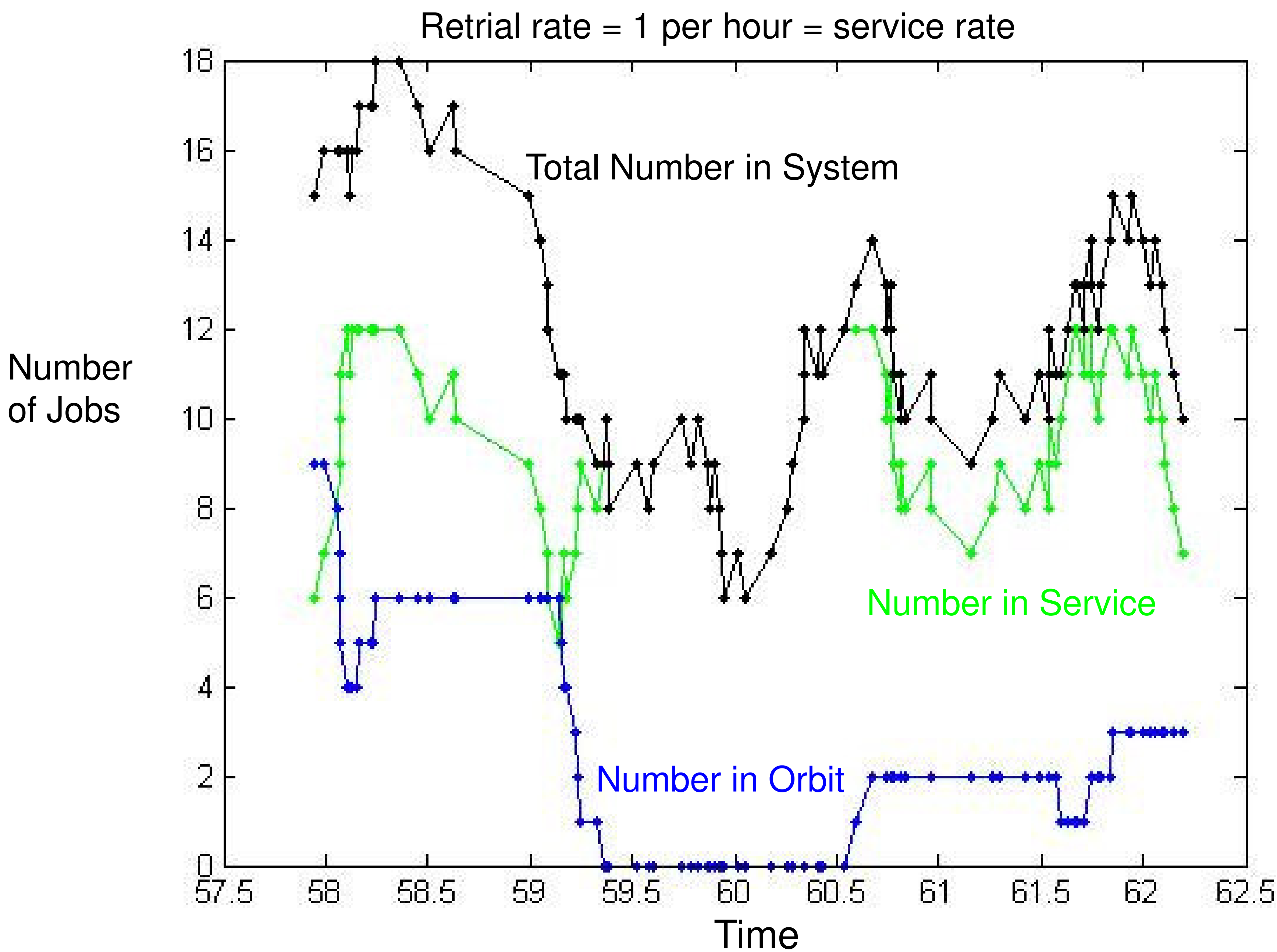
CanQueue 2009, Univ. of Windsor


ORBIT
If all servers are busy, incoming or retrying calls are bounced to orbit
Calls retry to enter service
Arrive
Servers
Depart

# Multiserver Retrial Queues

- Mobile phone: link to tower
- Satellite phone/data: link to satellite
- Dial-up internet
- Credit card verification

All have non-exponential retrials


Retrial rate = 1 per hour = service rate
Number of Jobs
Total Number in System
Number in Service
Number in Orbit
Time
0
2
4
6
8
10
12
14
16
18
57.5
58
58.5
59
59.5
60
60.5
61
61.5
62
62.5

# Single-Server Systems: Distribution matters!

- Ethernet and WiFi deliberately avoid using deterministic retrial distributions
- They are single-server systems, though
- Multi-server systems generally act differently for measures like probability of delay.

# Our Main Question

- When must you take the retrial distribution into account?

- Methods:
  - Discrete-event simulation
  - Markov-chain computation

# M/M/c/0 + G-retrials

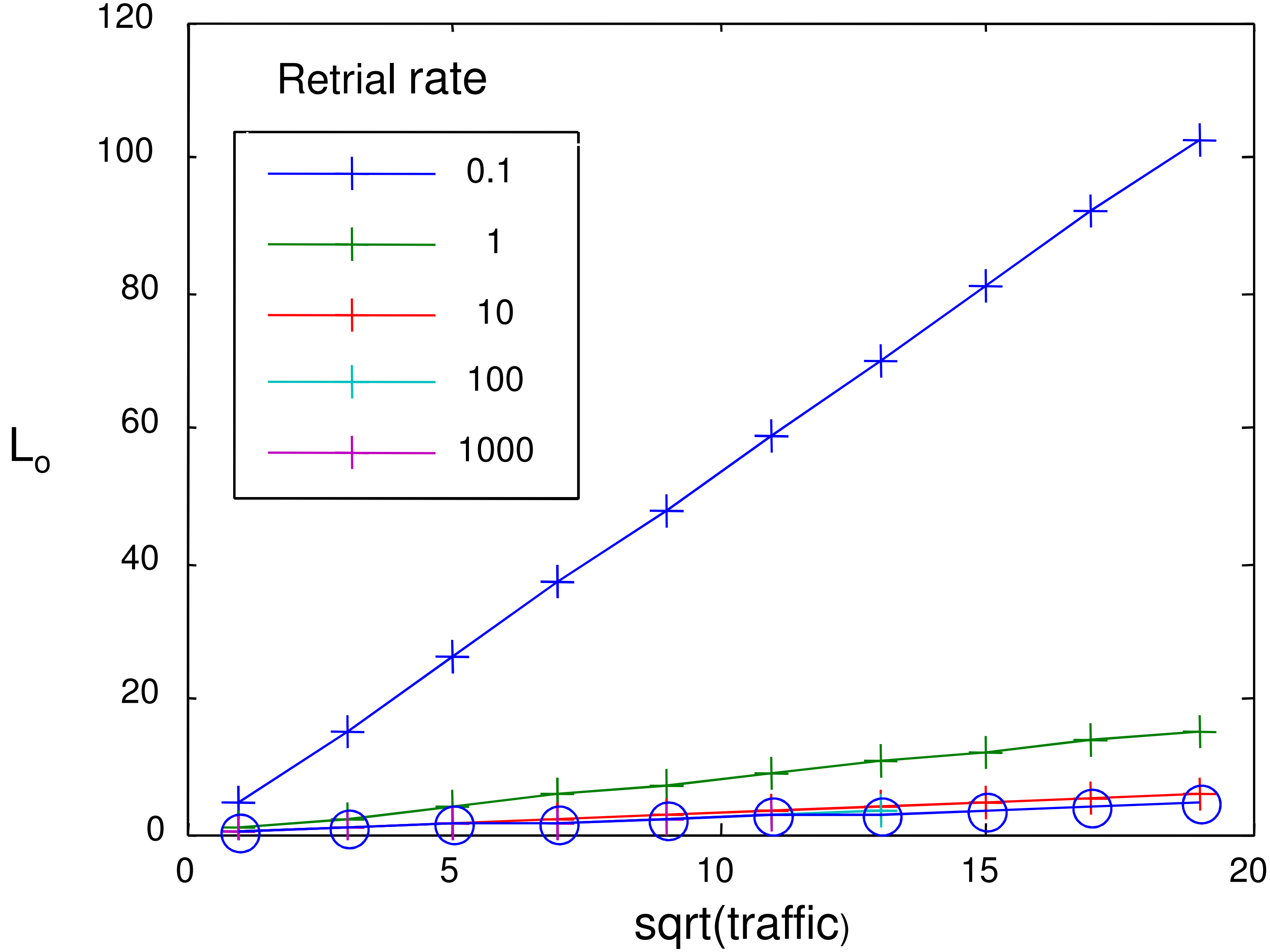
- Poisson arrivals with constant rate
- Exponential service
- No organized buffer
- Everyone in orbit retries
  - not just one person
- Customers never give up

# Square-Root Staffing

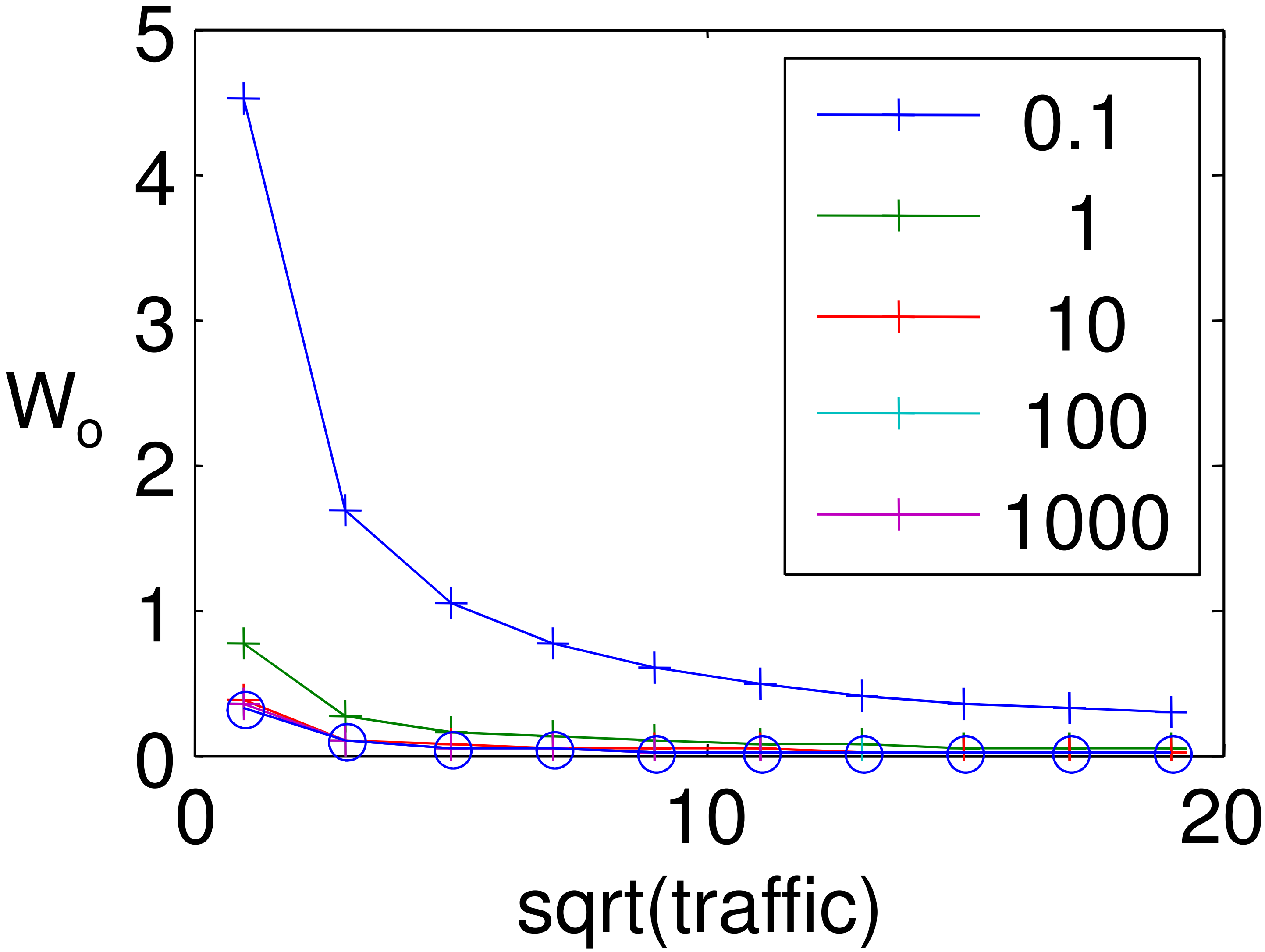
- Servers = traffic + 1*sqrt(traffic)

- QED: Quality and Efficiency Domain

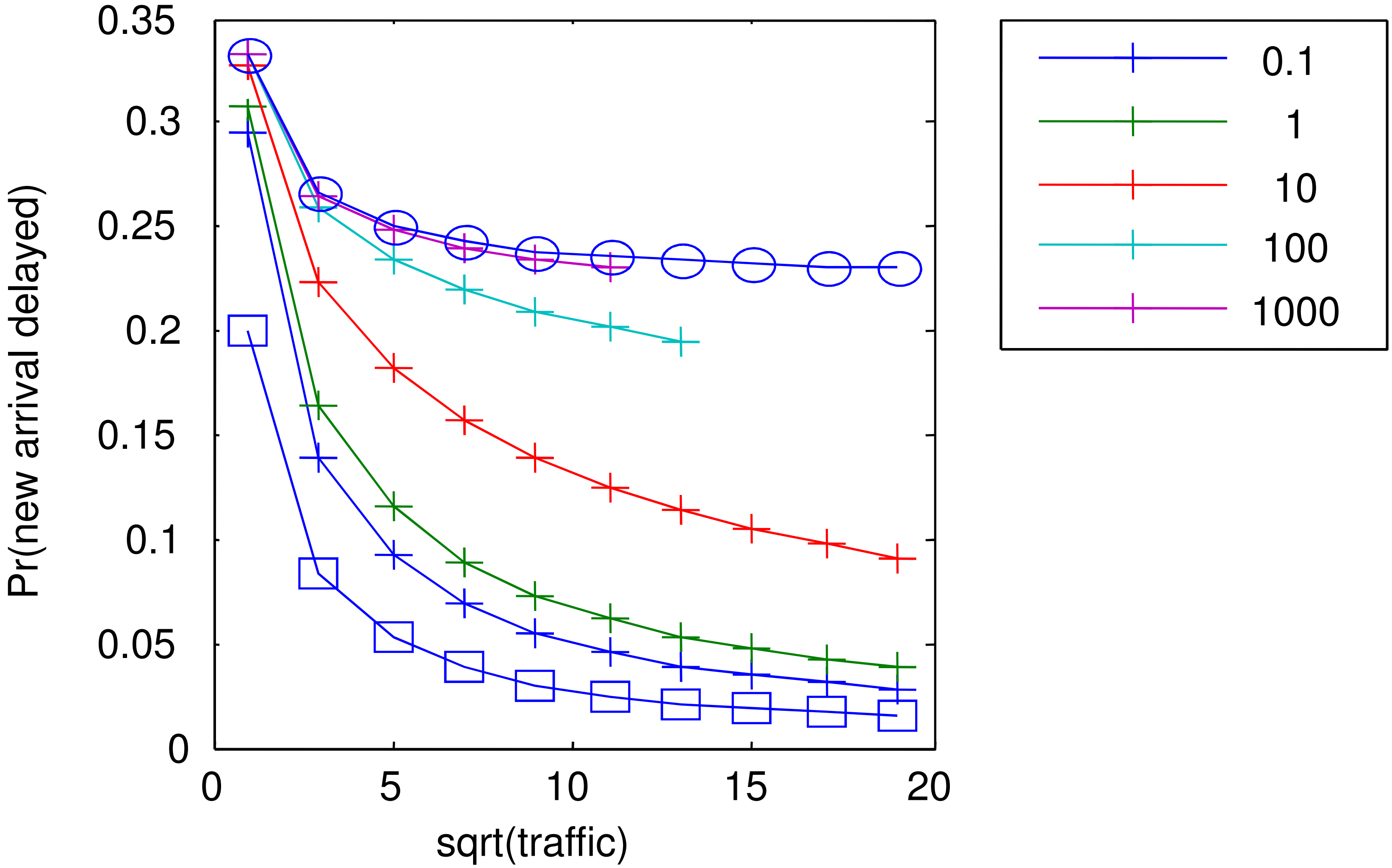
# $L_o$ as system grows



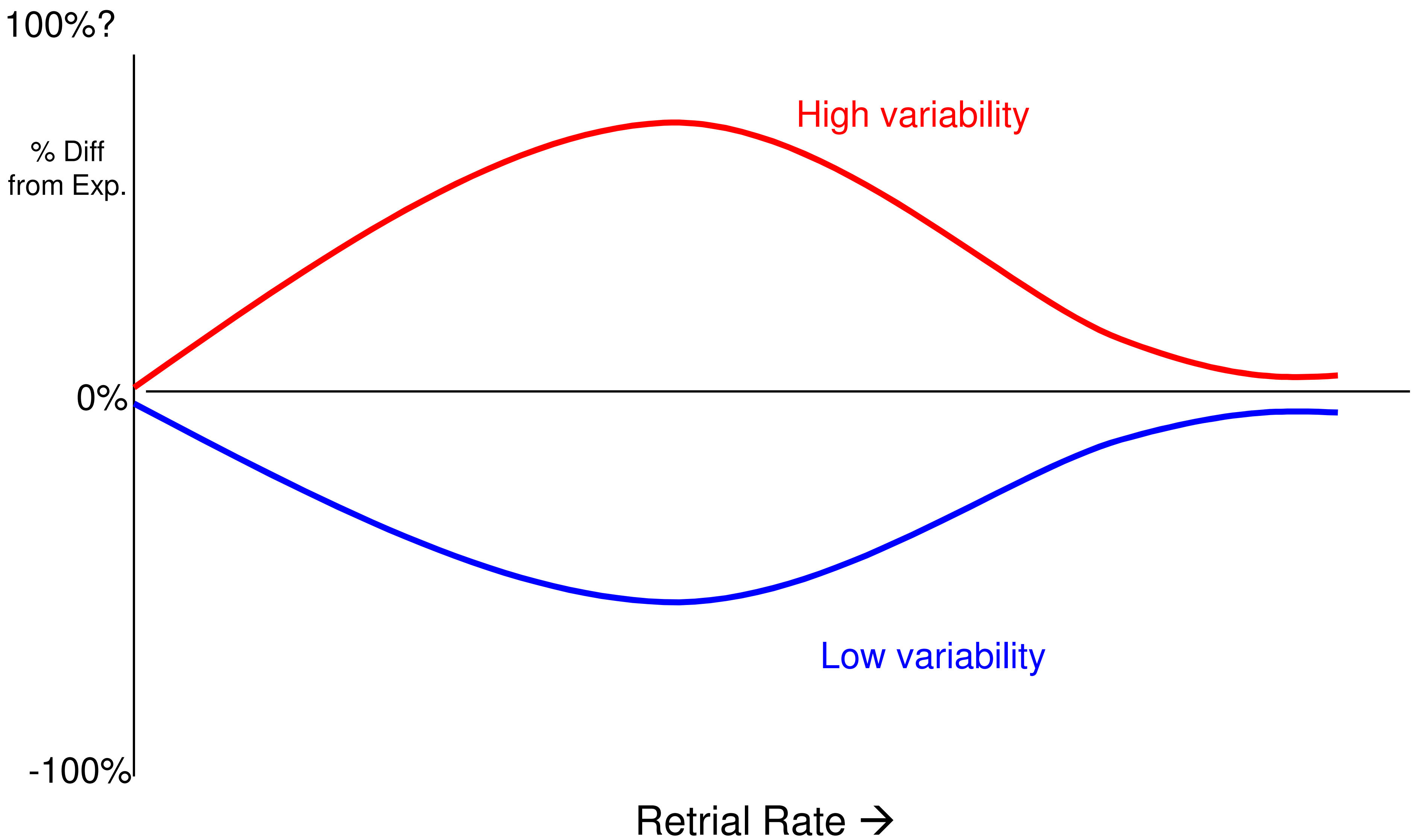
# $W_o$ as system grows



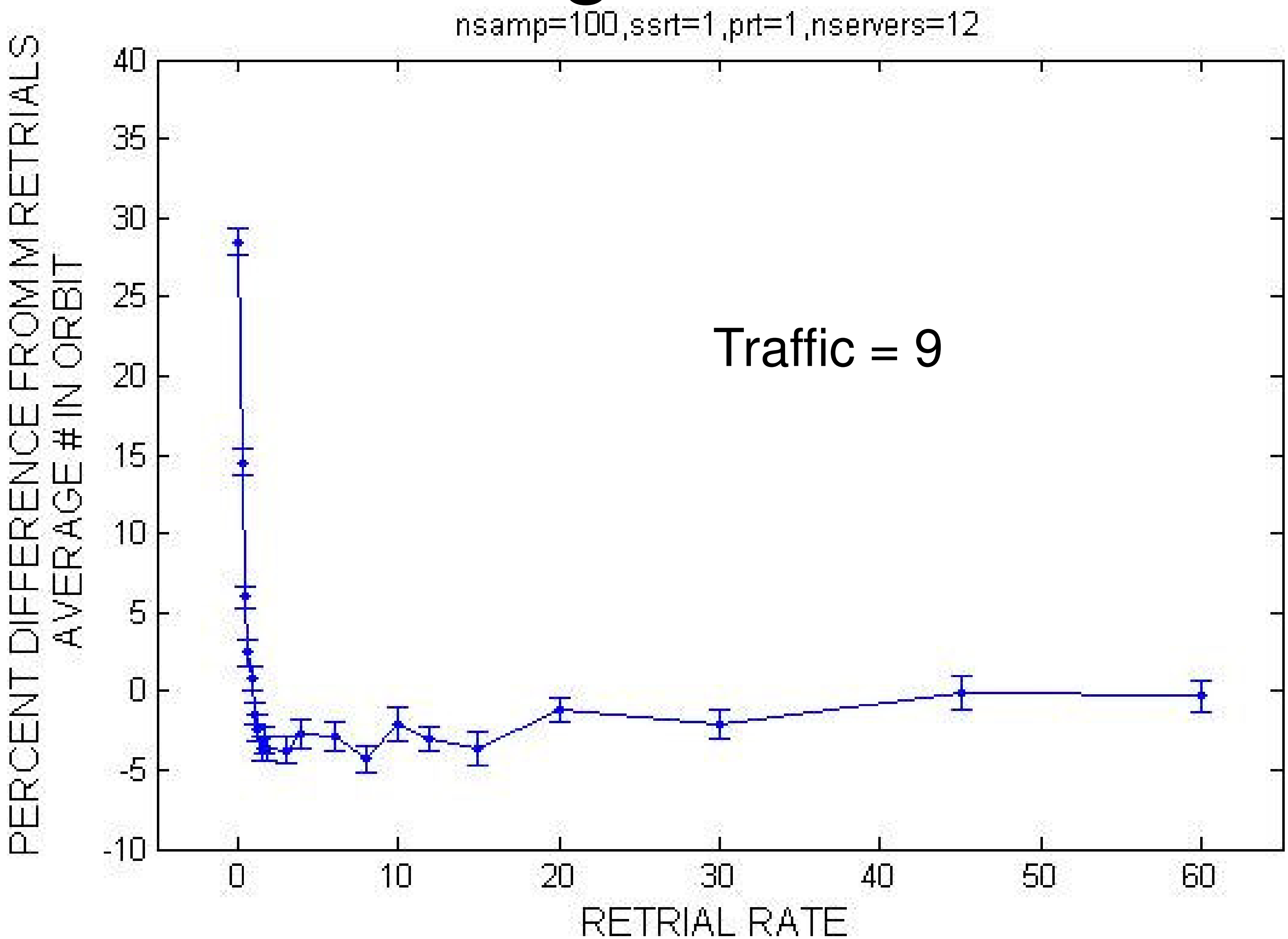
# Pr(new arrival delayed)



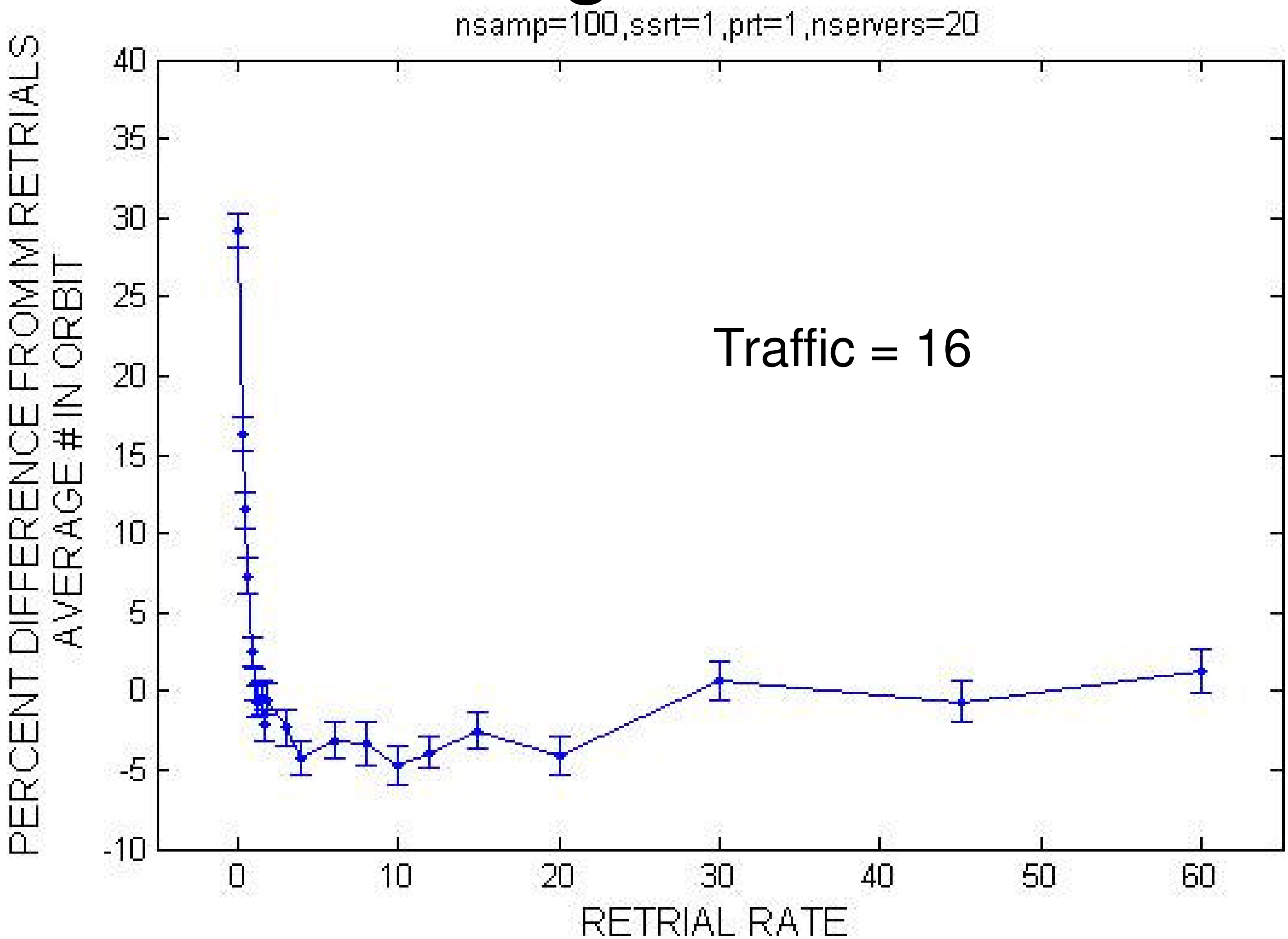
# What We Expected to See



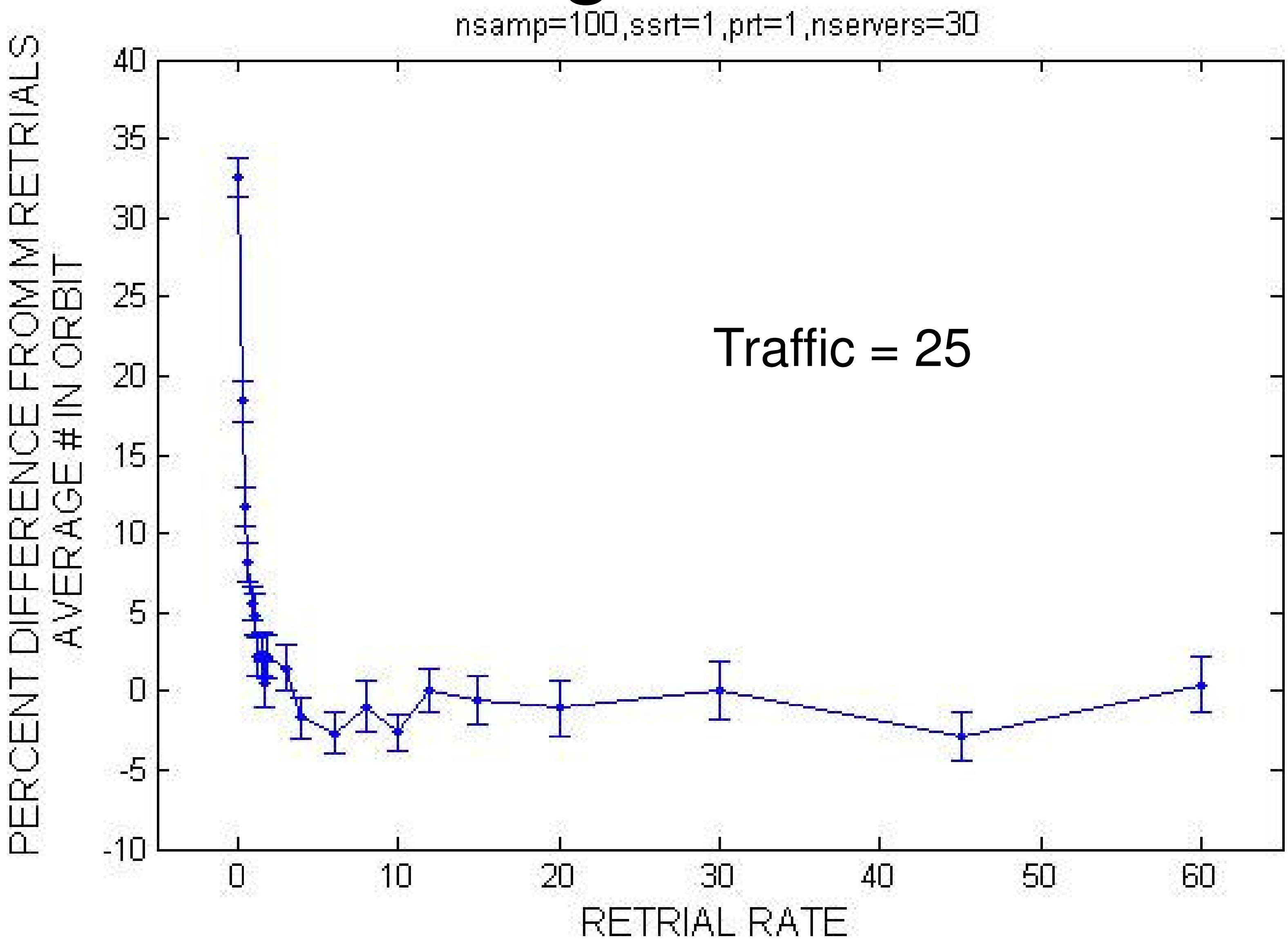
# % Diff: Average Number in Orbit



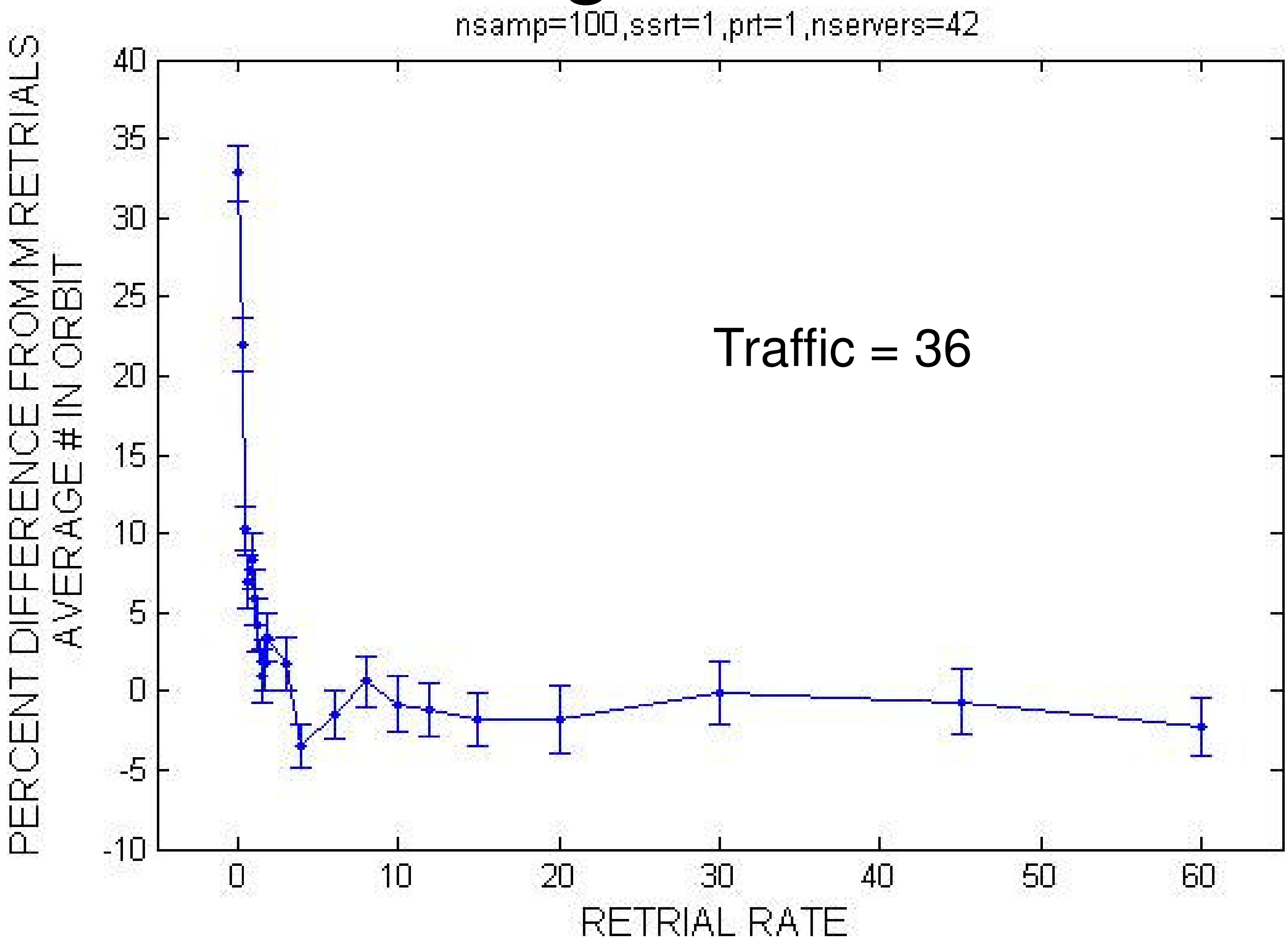
# % Diff: Average Number in Orbit



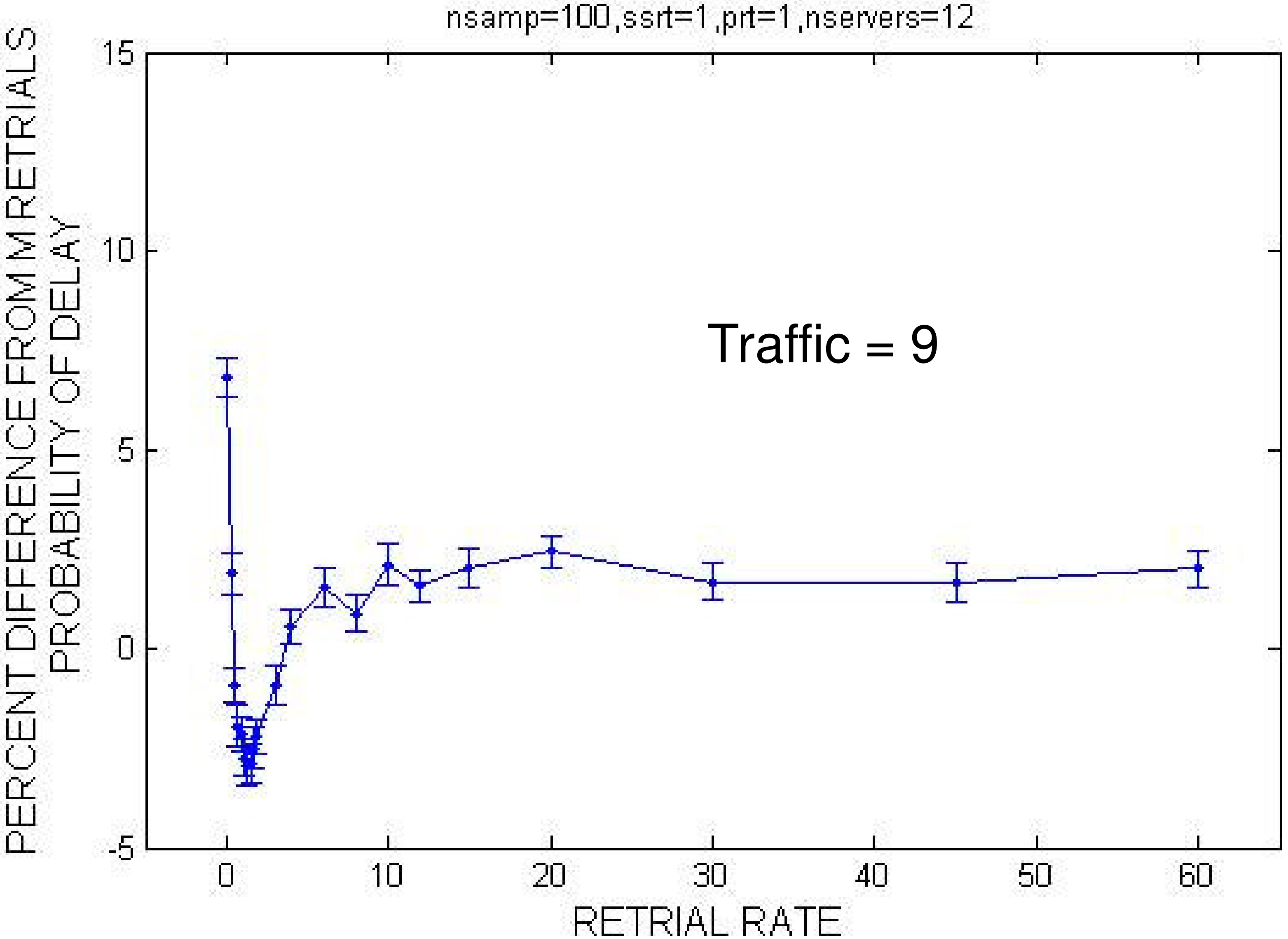
# % Diff: Average Number in Orbit



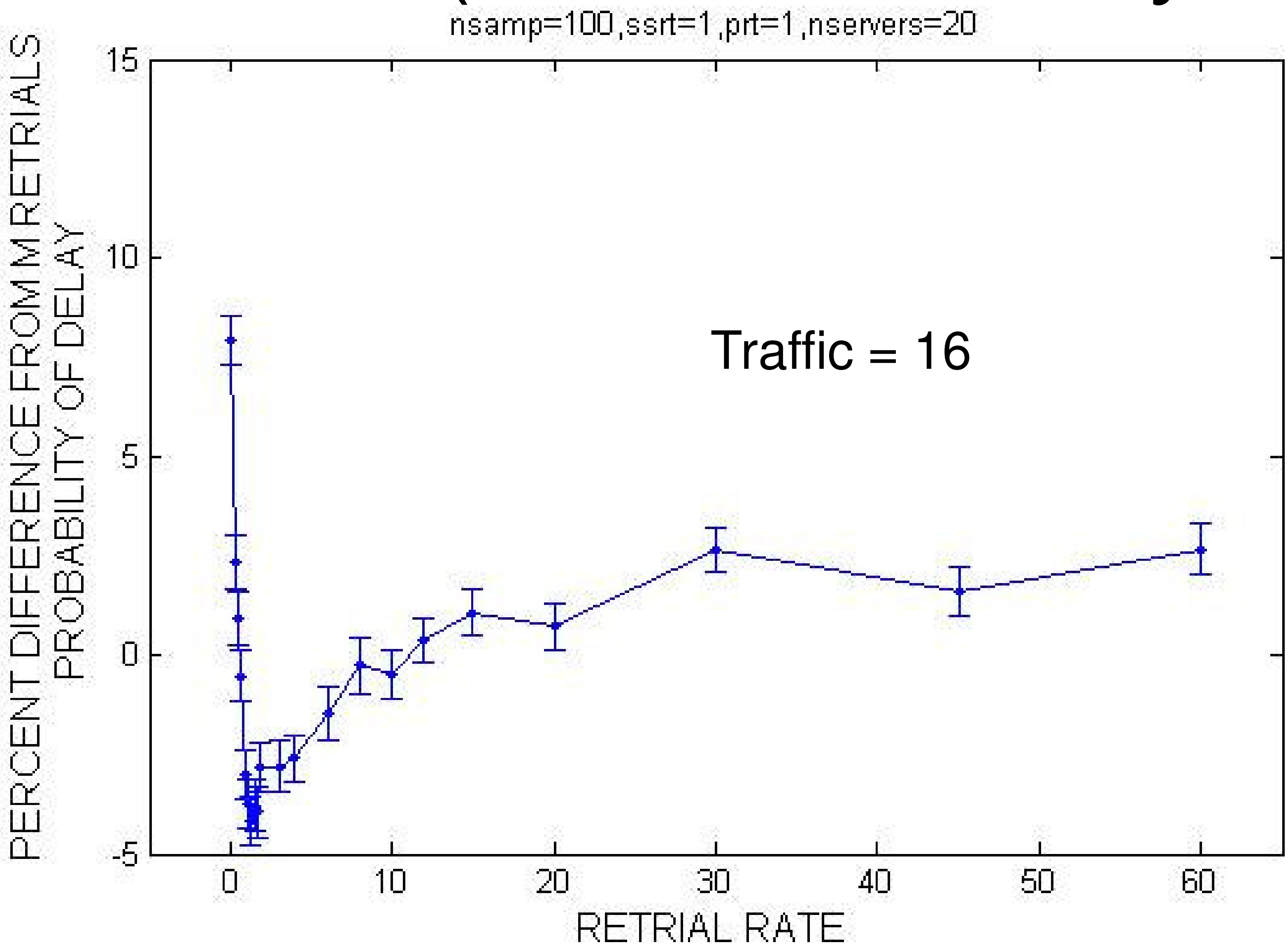
# % Diff: Average Number in Orbit



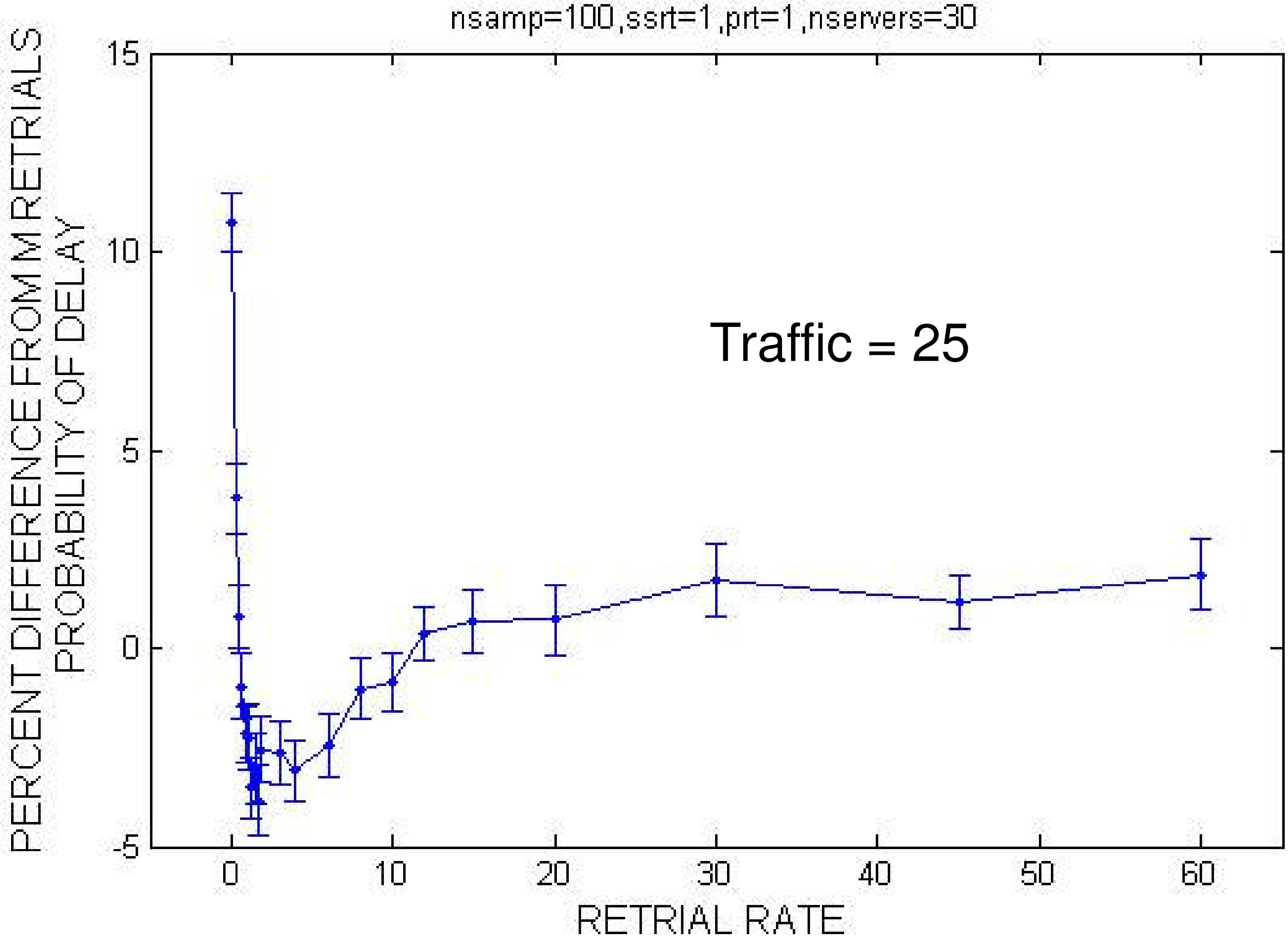
# % Diff: Pr(new arrival delayed)



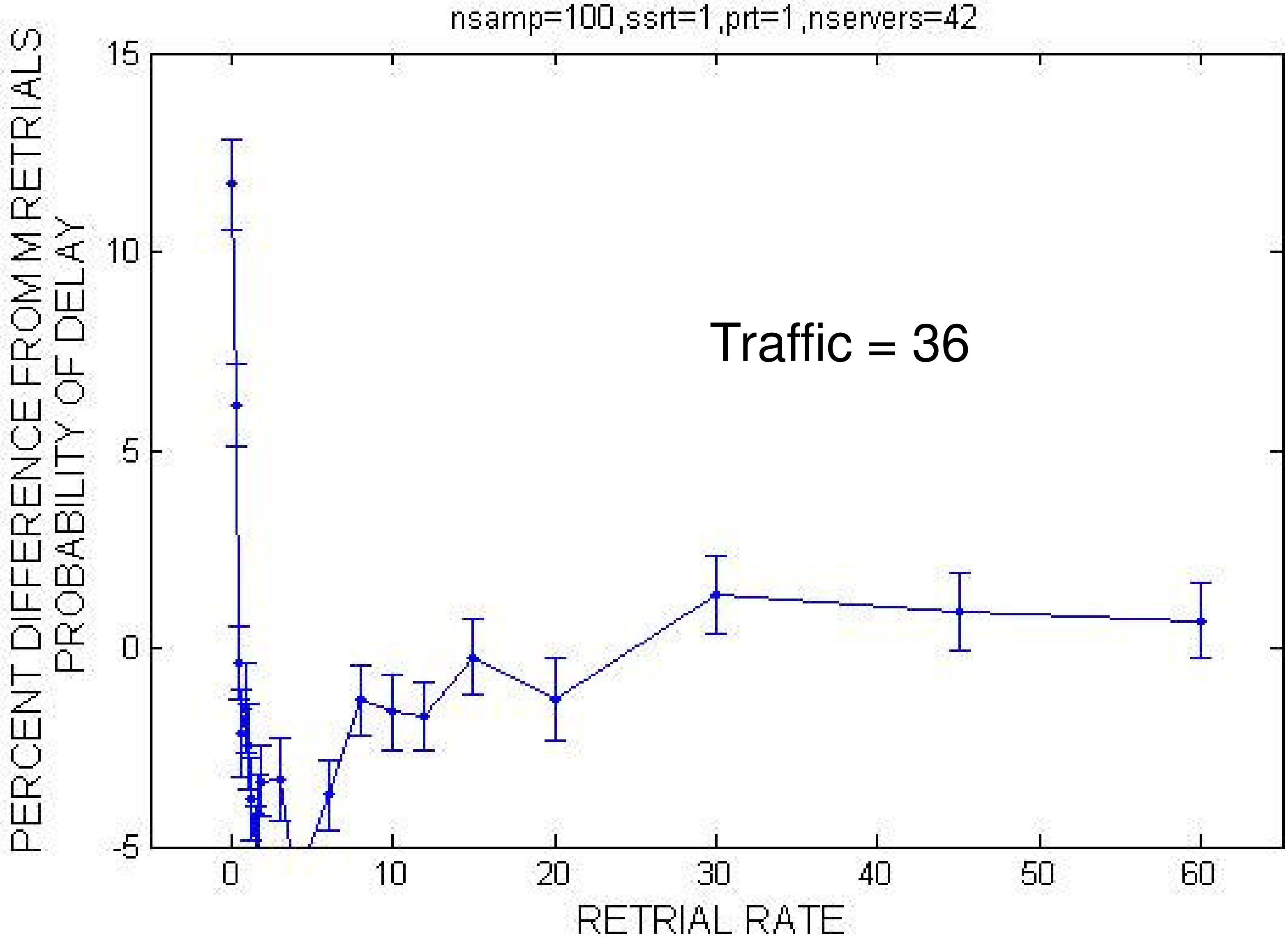
# % Diff: Pr(new arrival delayed)



# % Diff: Pr(new arrival delayed)



# % Diff: Pr(new arrival delayed)



WHY?

# Exponential Retrials

| | Retry 1 | Retry 2 | Retry 3 | Retry 4 |
|---|---|---|---|---|
| Job 1 | 10 | 27 | 4 | 22 |
| Job 2 | 19 | 11 | 23 | 5 |
| Job 3 | 7 | 51 | 13 | 17 |

# Personal Retrial Times (PRT)

| | Retry 1 | Retry 2 | Retry 3 | Retry 4 |
|---|---|---|---|---|
| Job 1 | 10 | 10 | 10 | 10 |
| Job 2 | 19 | 19 | 19 | 19 |
| Job 3 | 7 | 7 | 7 | 7 |

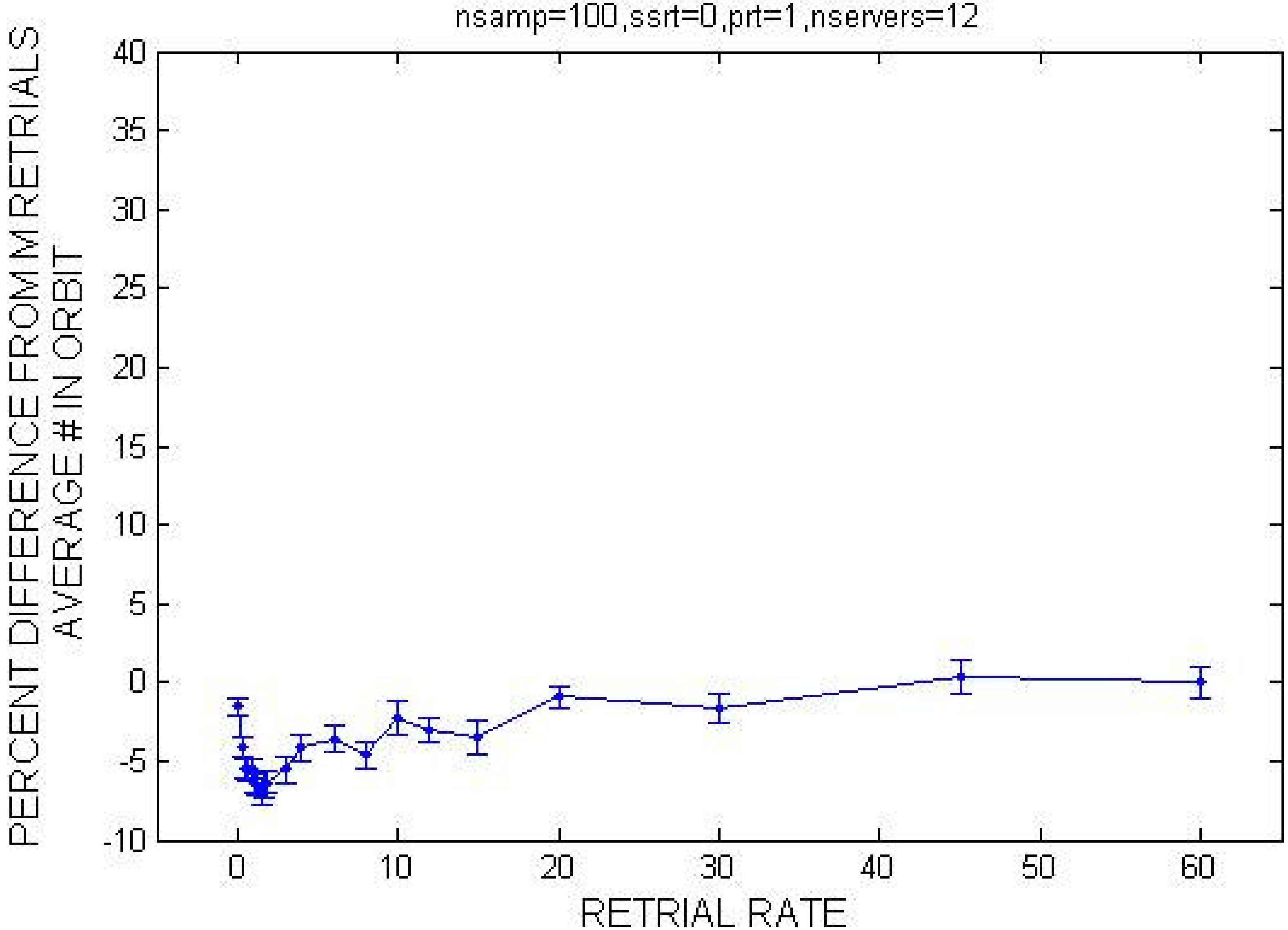
# Shared Sequence of Retrial Times (SSRT)

| | Retry 1 | Retry 2 | Retry 3 | Retry 4 |
|---|---|---|---|---|
| Job 1 | 10 | 27 | 4 | 22 |
| Job 2 | 10 | 27 | 4 | 22 |
| Job 3 | 10 | 27 | 4 | 22 |

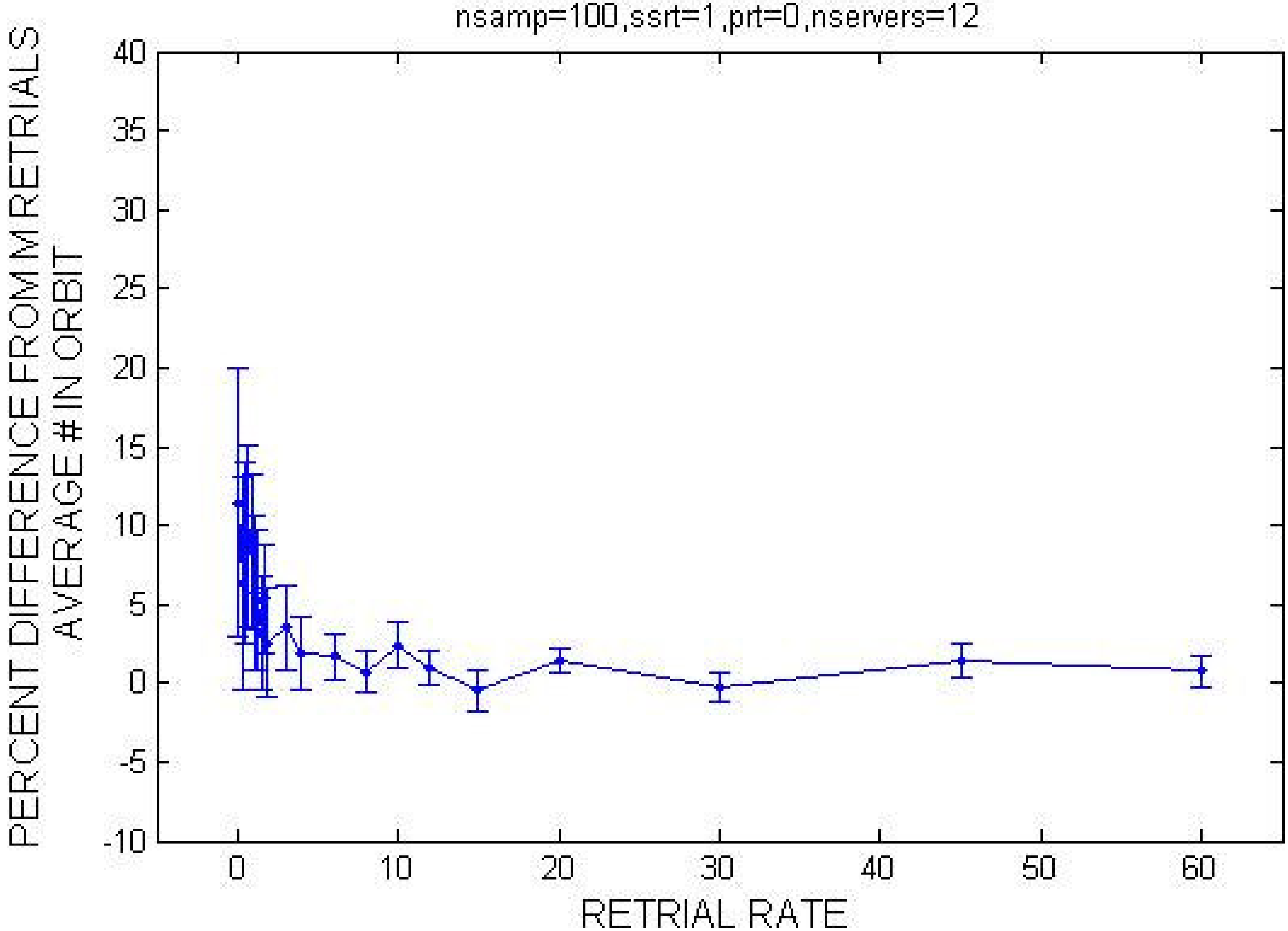
# Deterministic Retrials

| | Retry 1 | Retry 2 | Retry 3 | Retry 4 |
|---|---|---|---|---|
| Job 1 | 10 | 10 | 10 | 10 |
| Job 2 | 10 | 10 | 10 | 10 |
| Job 3 | 10 | 10 | 10 | 10 |

# % Diff in Lo: PRT vs M

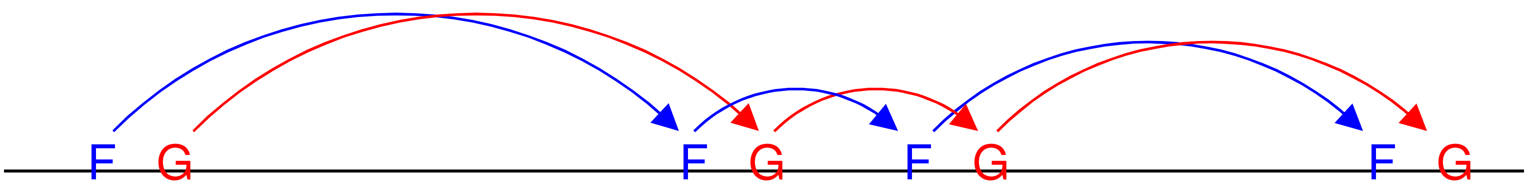


# % Diff in Lo: SSRT vs M



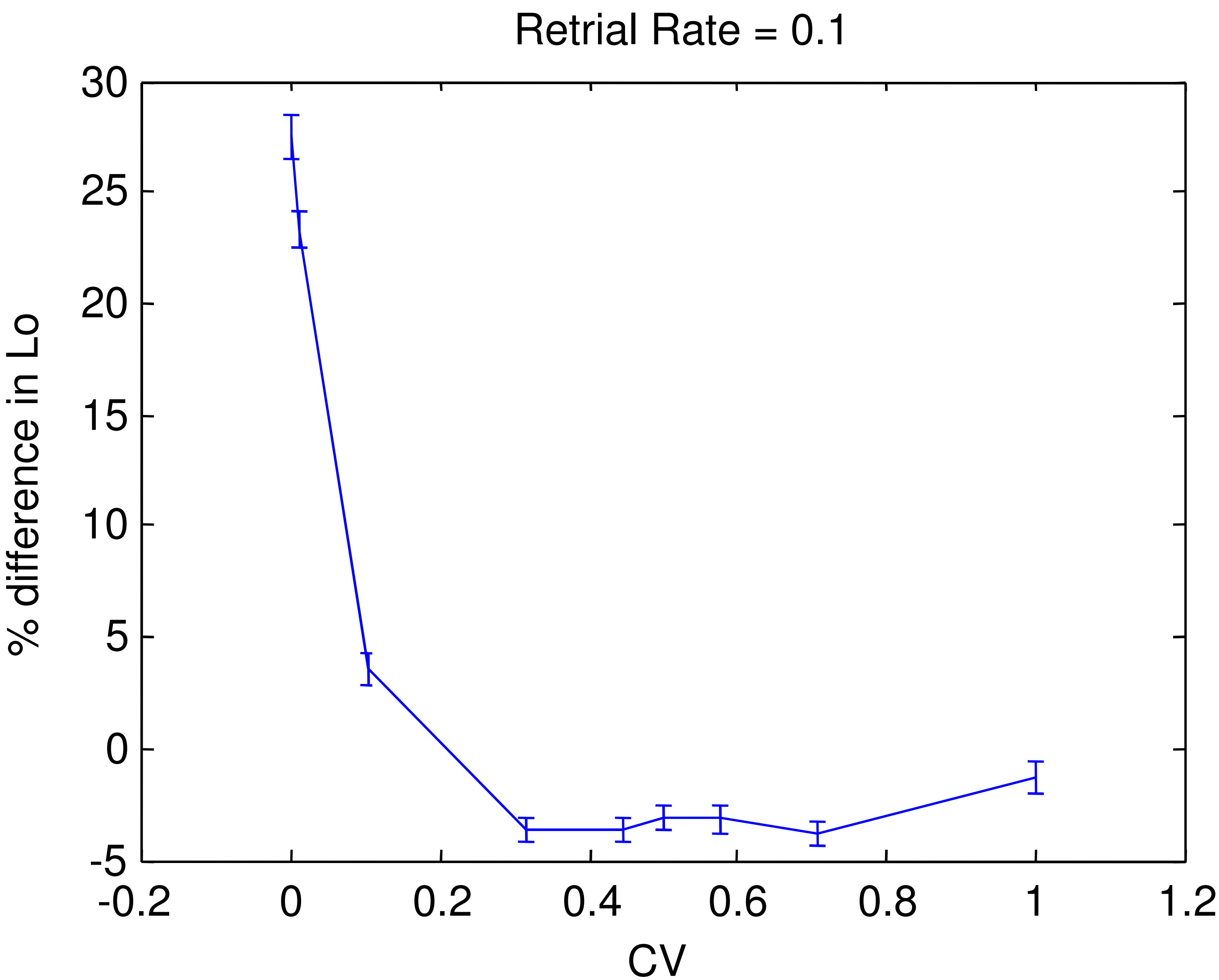
# Why? Because:

- Shared Sequence of Retrial Times is the dominant effect.



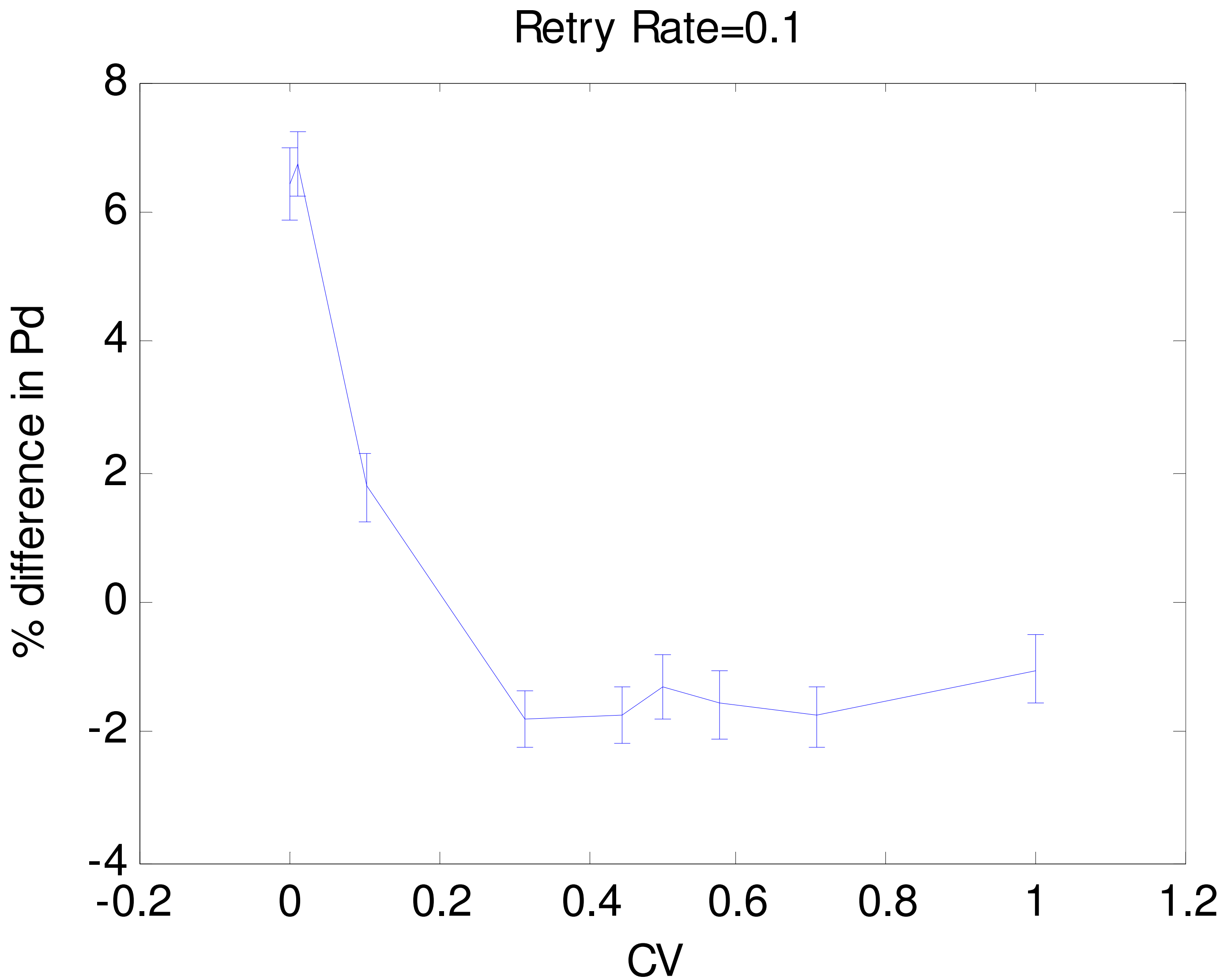

- How deterministic does it have to be?
- We will change the Coefficient of Variation (CV)

# % Diff in Lo



Retrial Rate = 0.1
30
25
20
15
10
5
0
-5
% difference in Lo
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
CV

# % Diff in Pr(delay)



# Markovian Approach

- $M/M/c/0 + PH_2$ retrials

- Lower limit on variability:
  - Two-phase Erlang has Squared Coefficient of Variation = 1/2

- Can get lower SCV using negative(!) probabilities

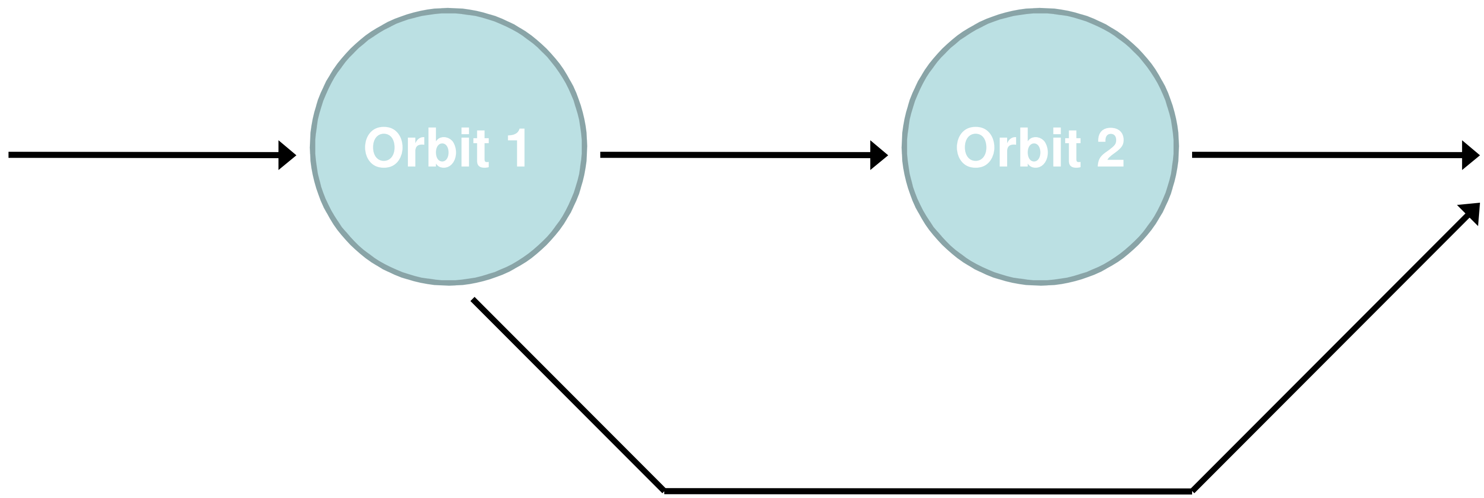
# Extended Probabilities

- 1955, Cox: Complex probabilities
- 1987, Nojo and Watanabe:

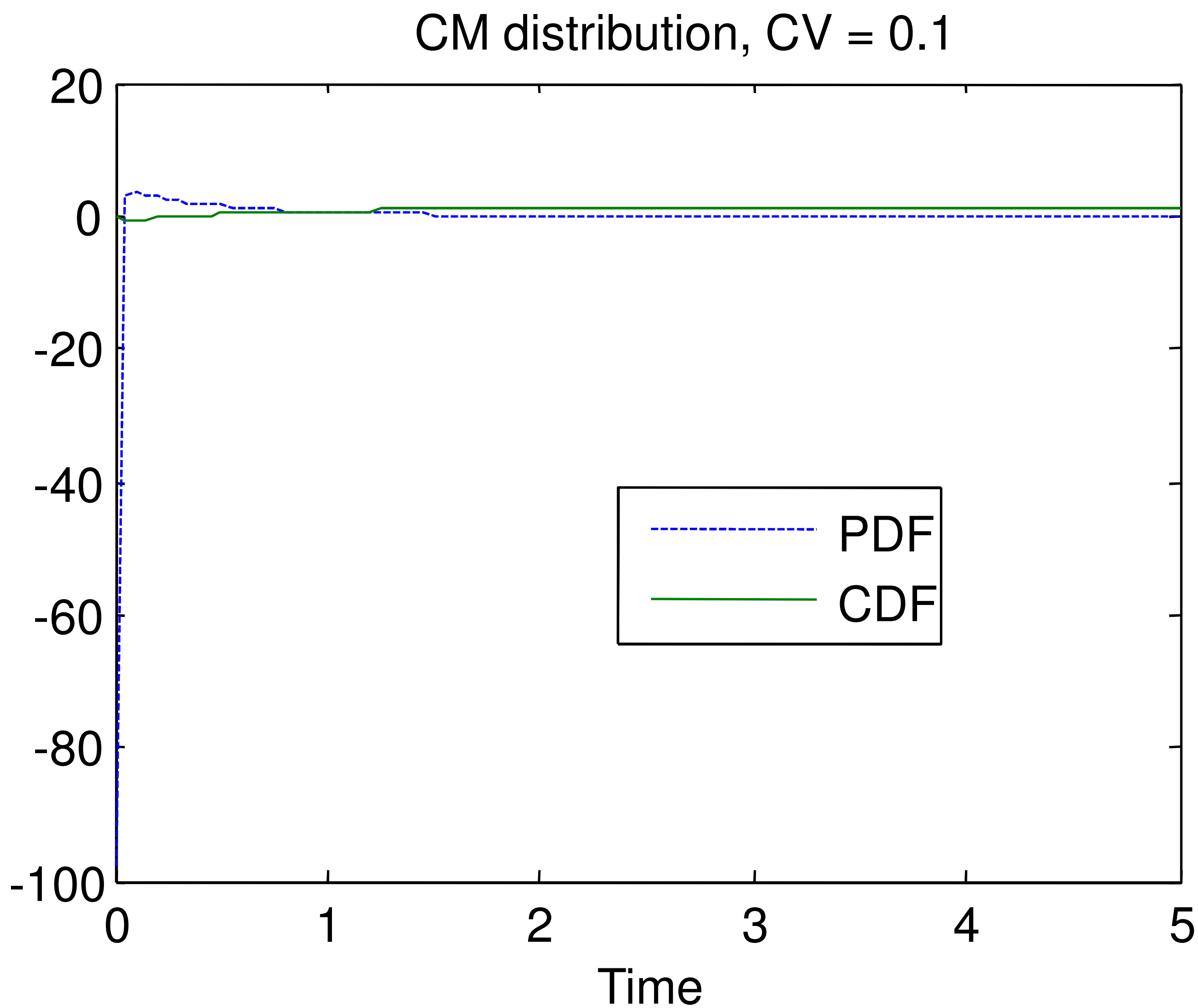
  Negative branching Probability (NP) distrib.
- 1994, Graham, Knuth, Patashnik
- 1999, Ball et al.:

  $H_2^*$ distribution
- 2007/8, Tijms: M/D/1 via $M/PH_2/1$
- Quantum physics

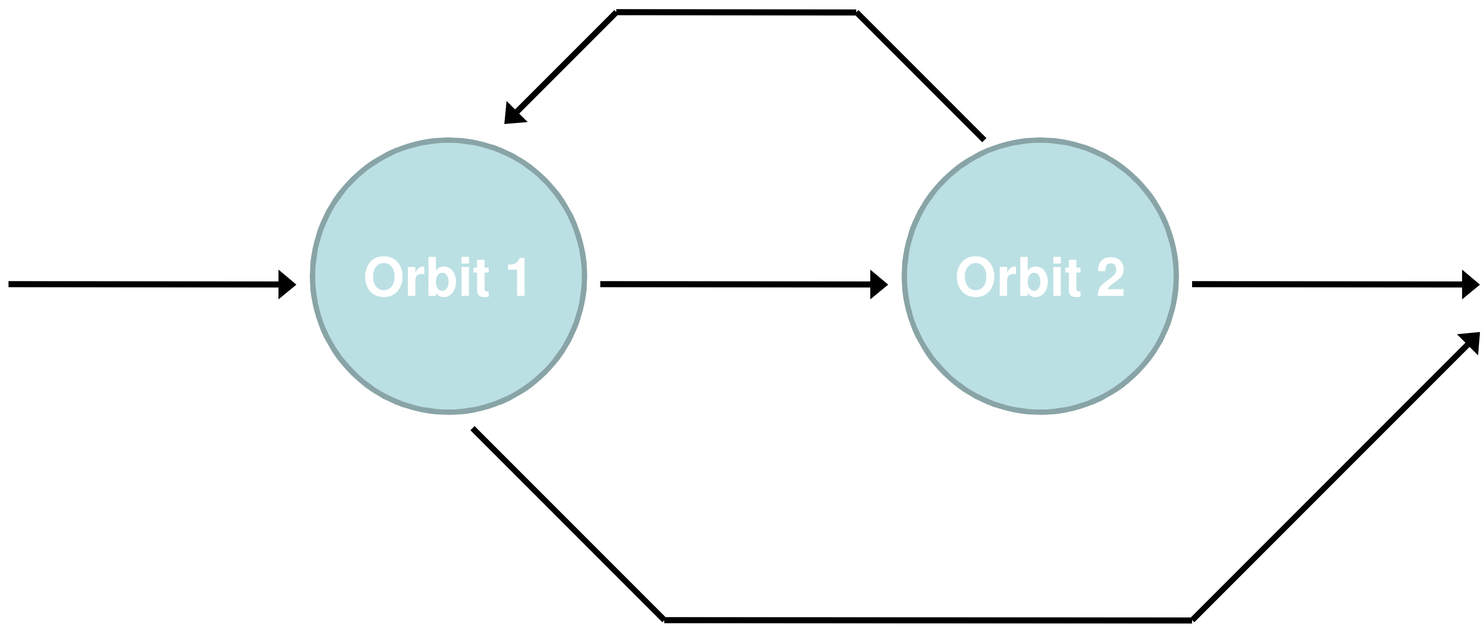
# Cox-Marie distribution



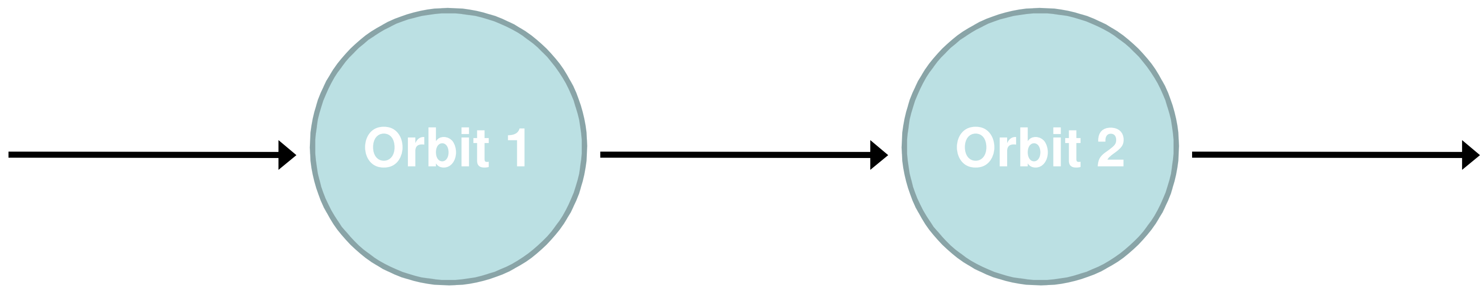
# Cox-Marie distribution



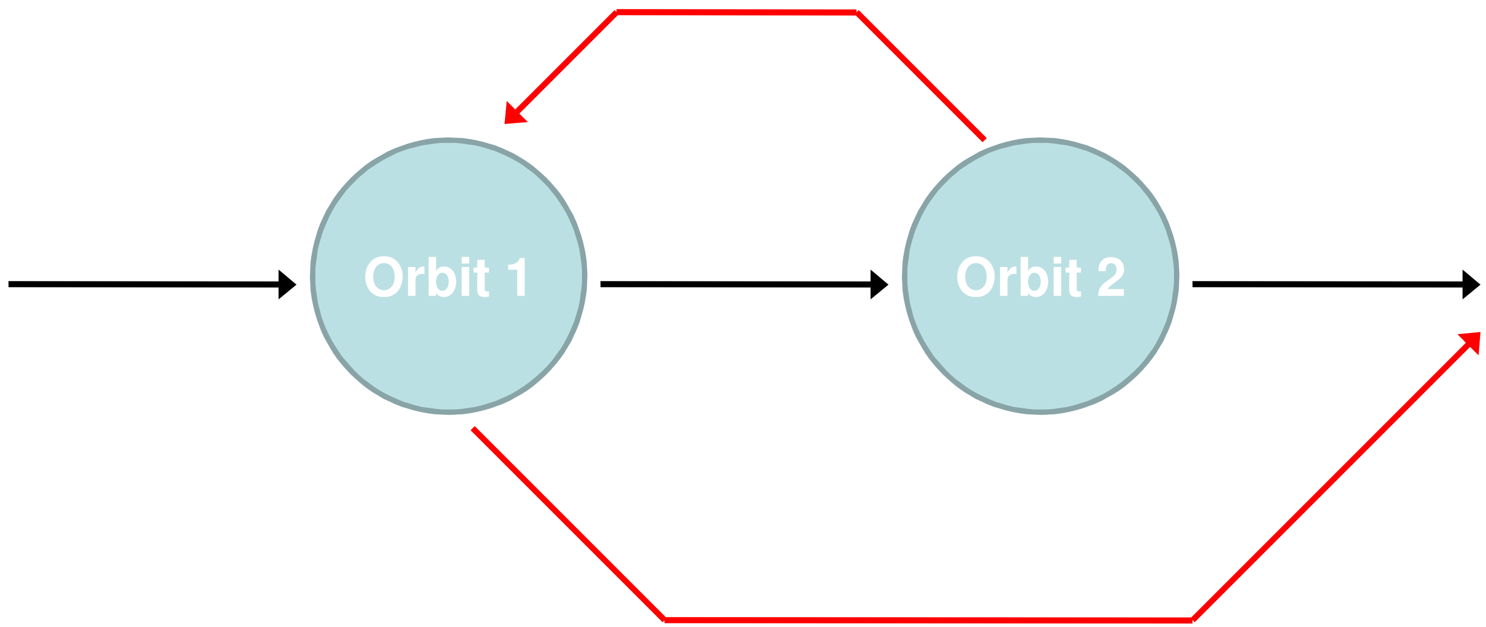
# NP distribution, SCV > 1/2



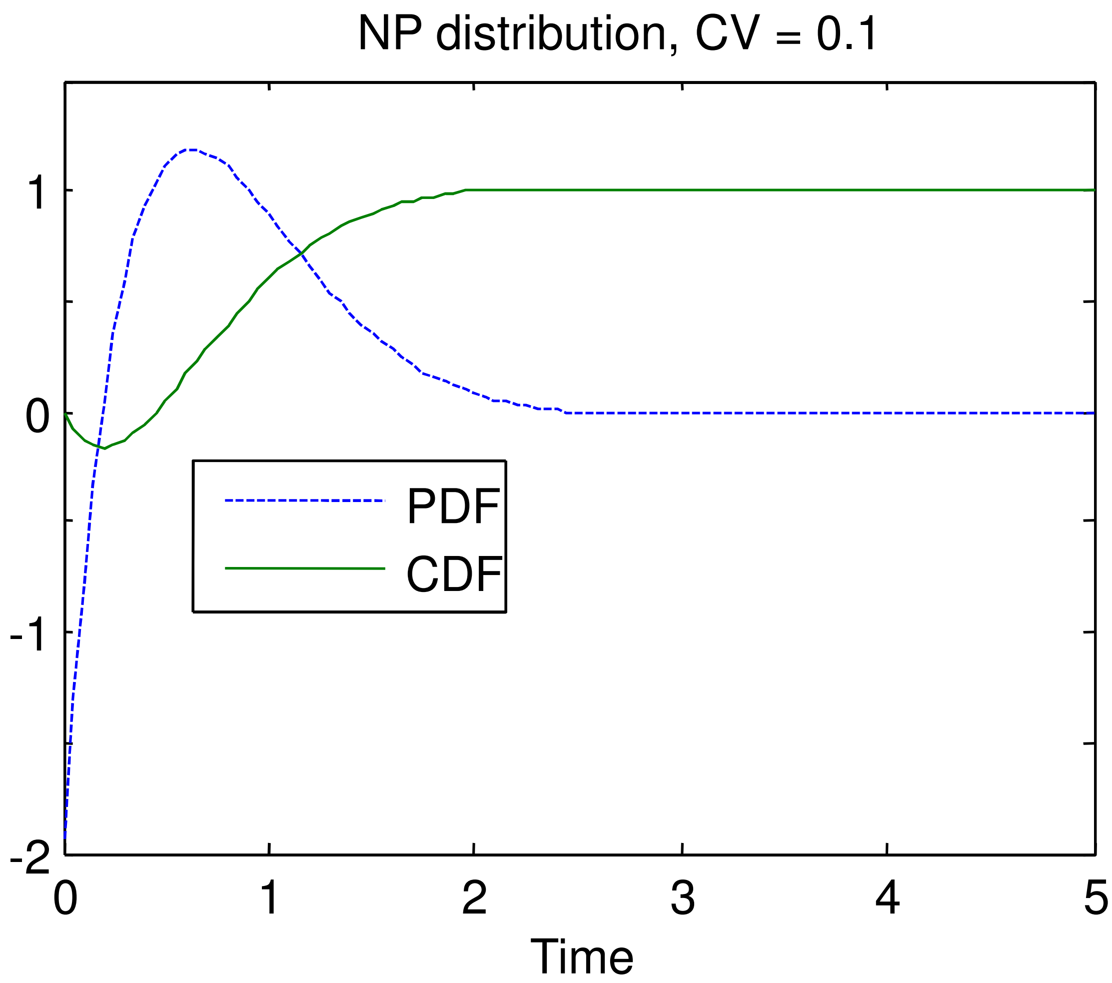
# NP distribution, SCV = 1/2



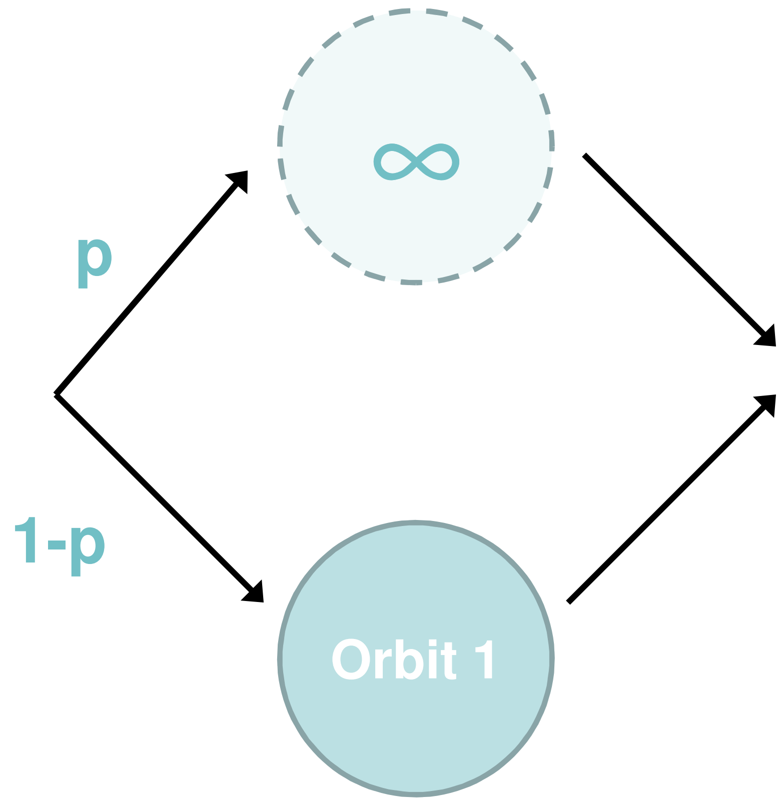
# NP distribution, SCV < 1/2



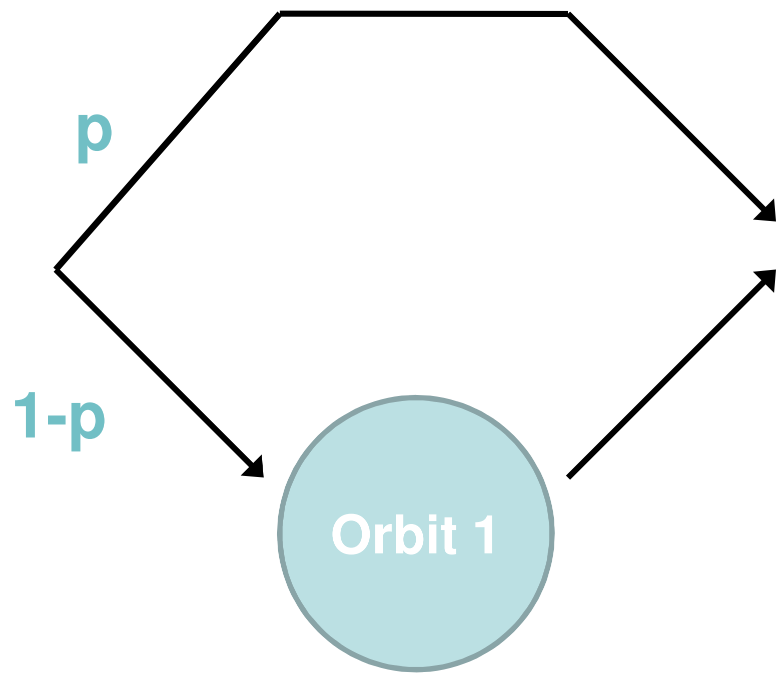
# NP Distribution



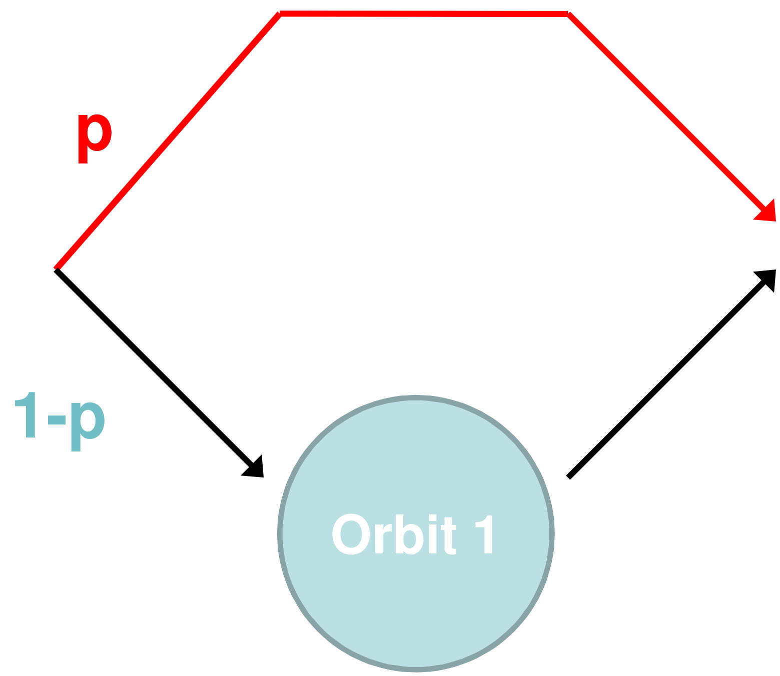
# $H_2^*$ distribution



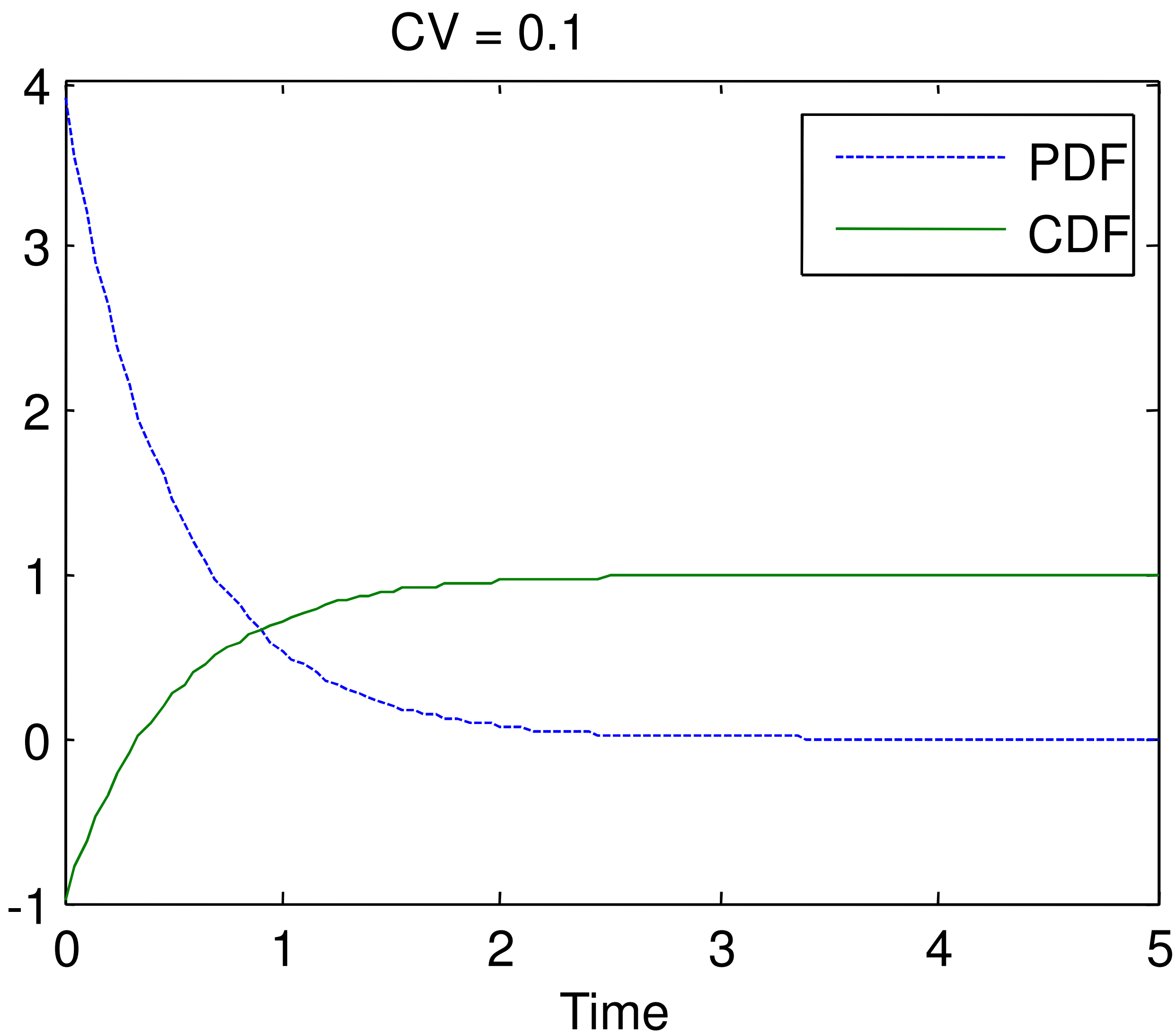
# $H_2^*$ distribution



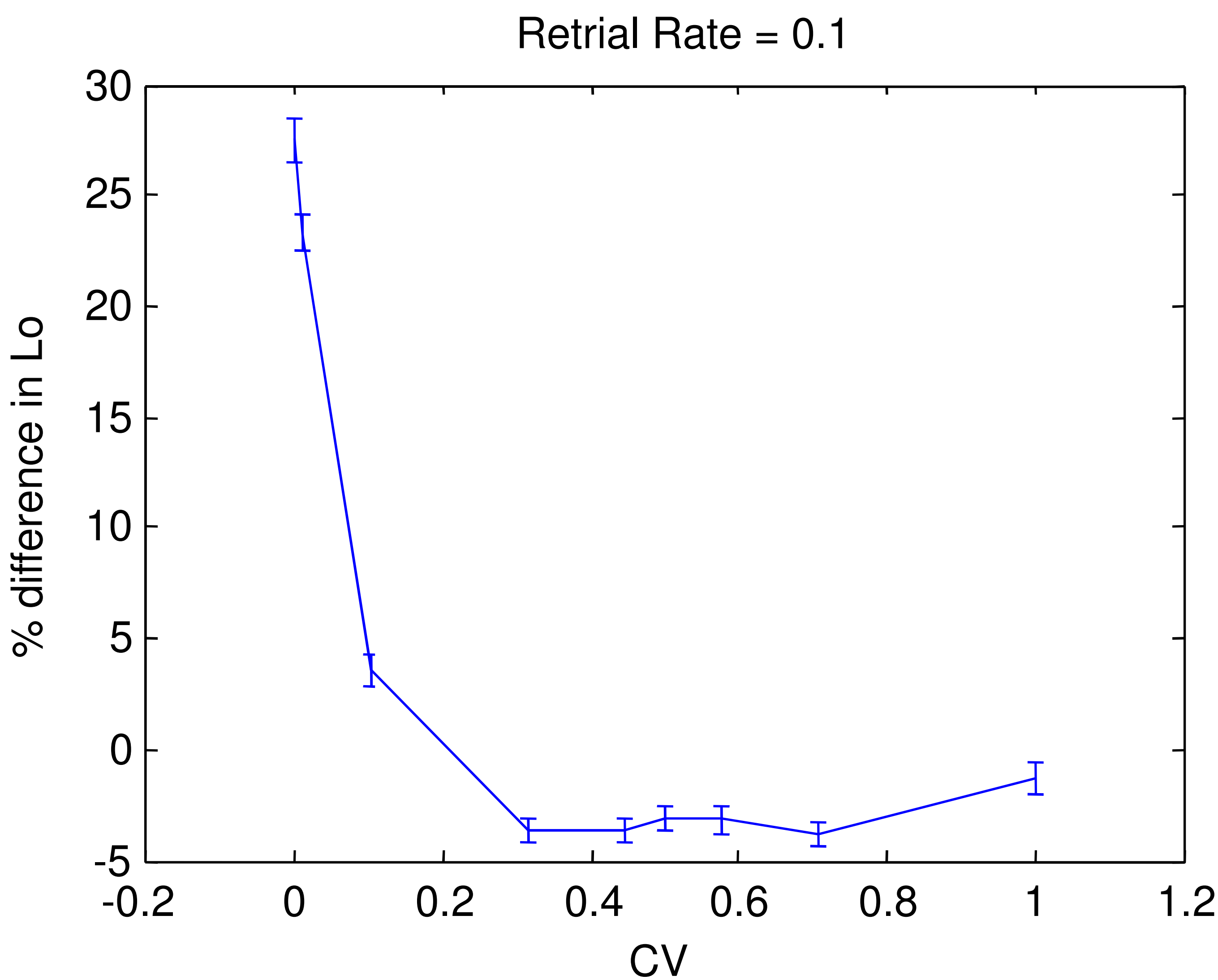
# $H_2$* distribution, SCV < 1

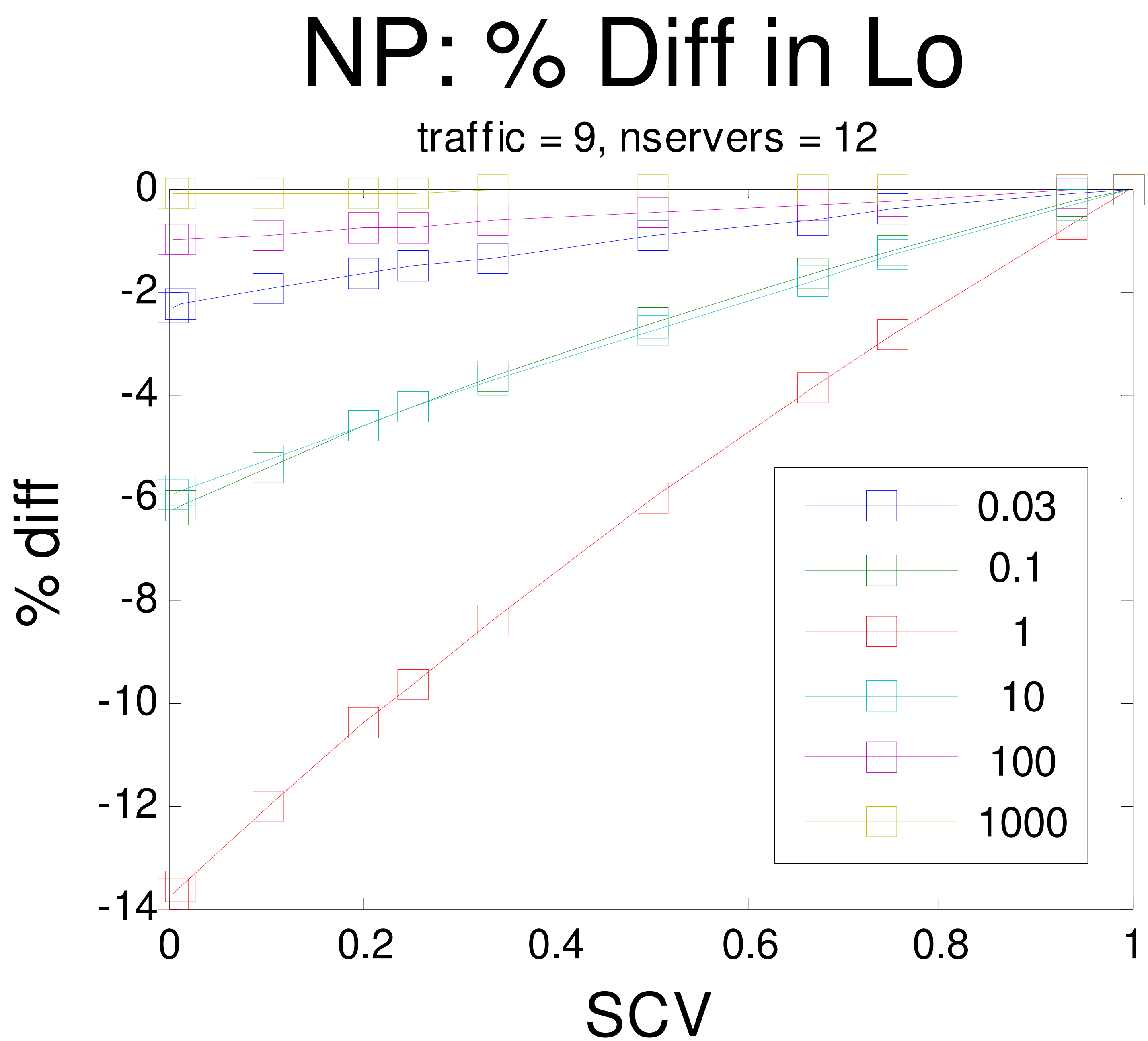


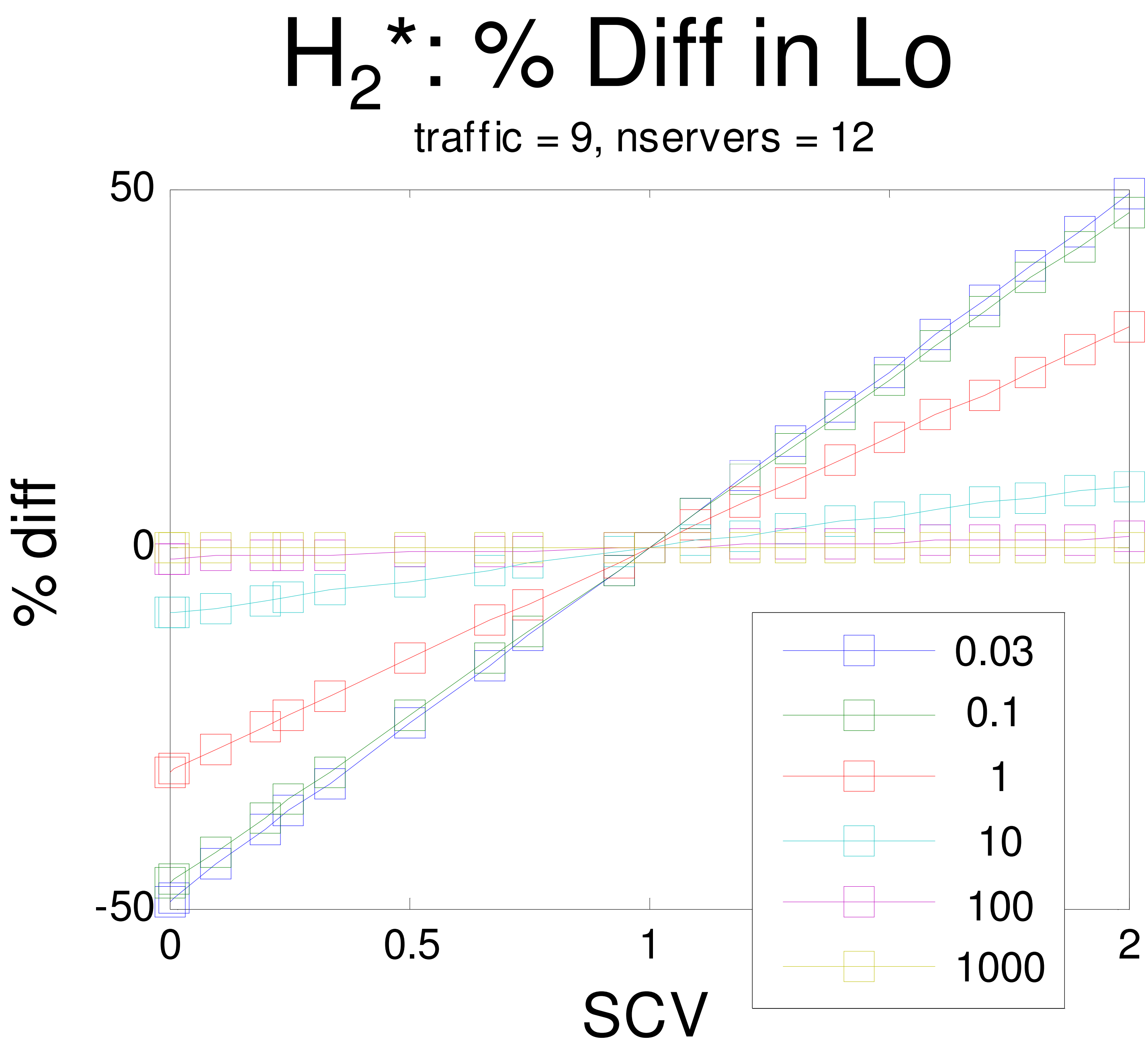
# $H_2$* distribution



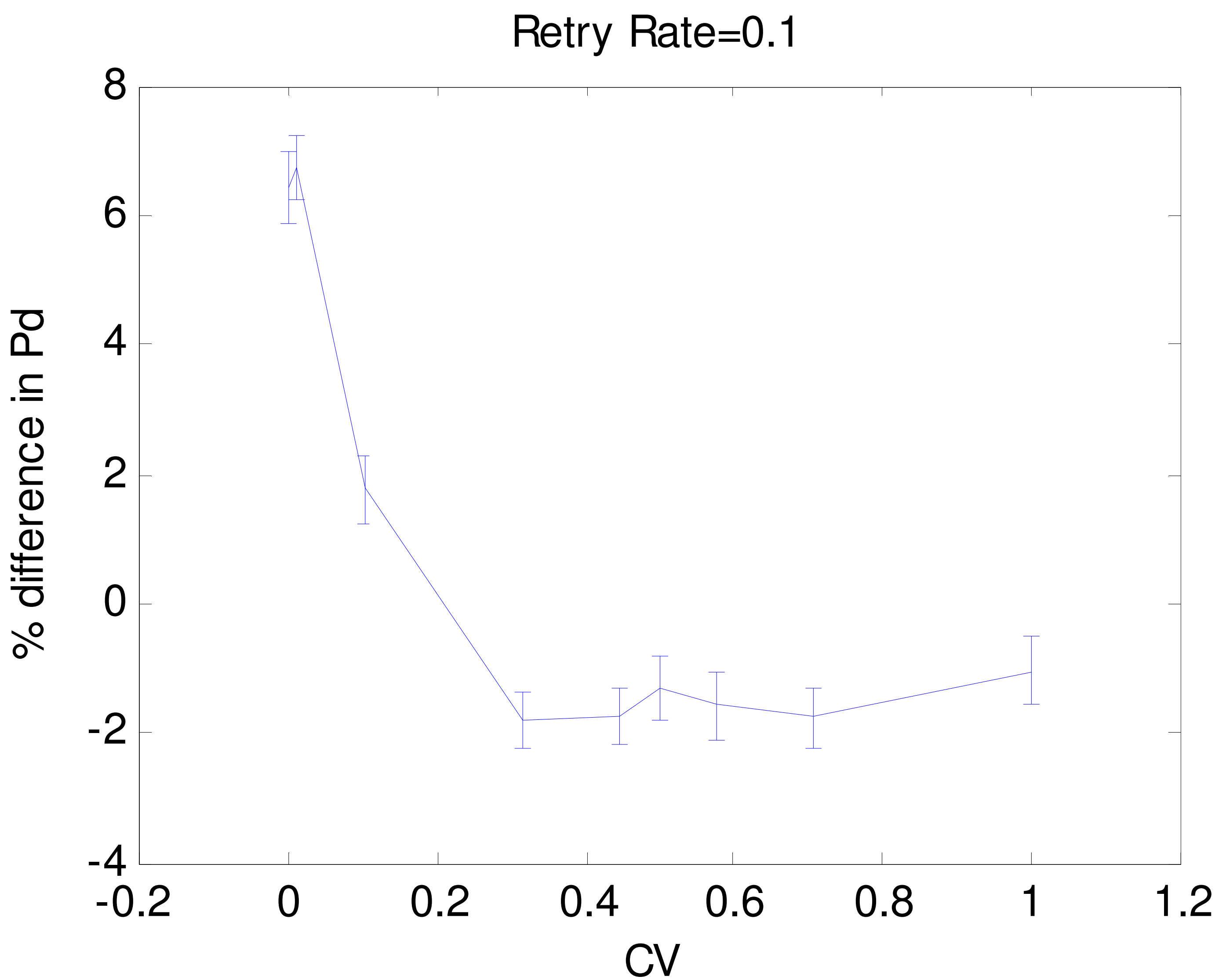
# Recall our simulations: Lo



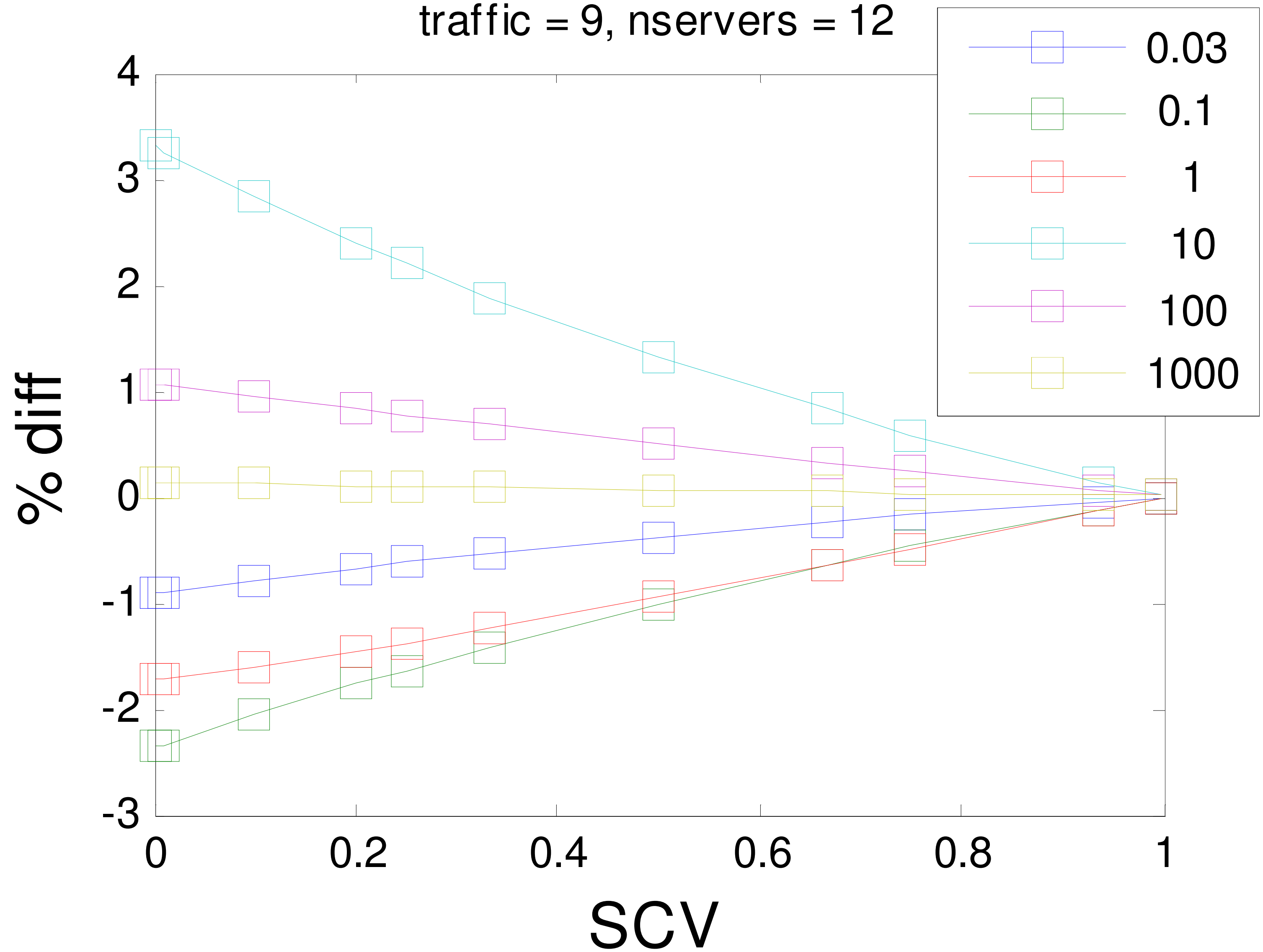

NP: % Diff in Lo
traffic = 9, nservers = 12
% diff
SCV
0
-2
-4
-6
-8
-10
-12
-14
0
0.2
0.4
0.6
0.8
1
0.03
0.1
1
10
100
1000


$H_2$*: % Diff in Lo
traffic = 9, nservers = 12
50
0
-50
% diff
0
0.5
1
2
SCV
0.03
0.1
1
10
100
1000

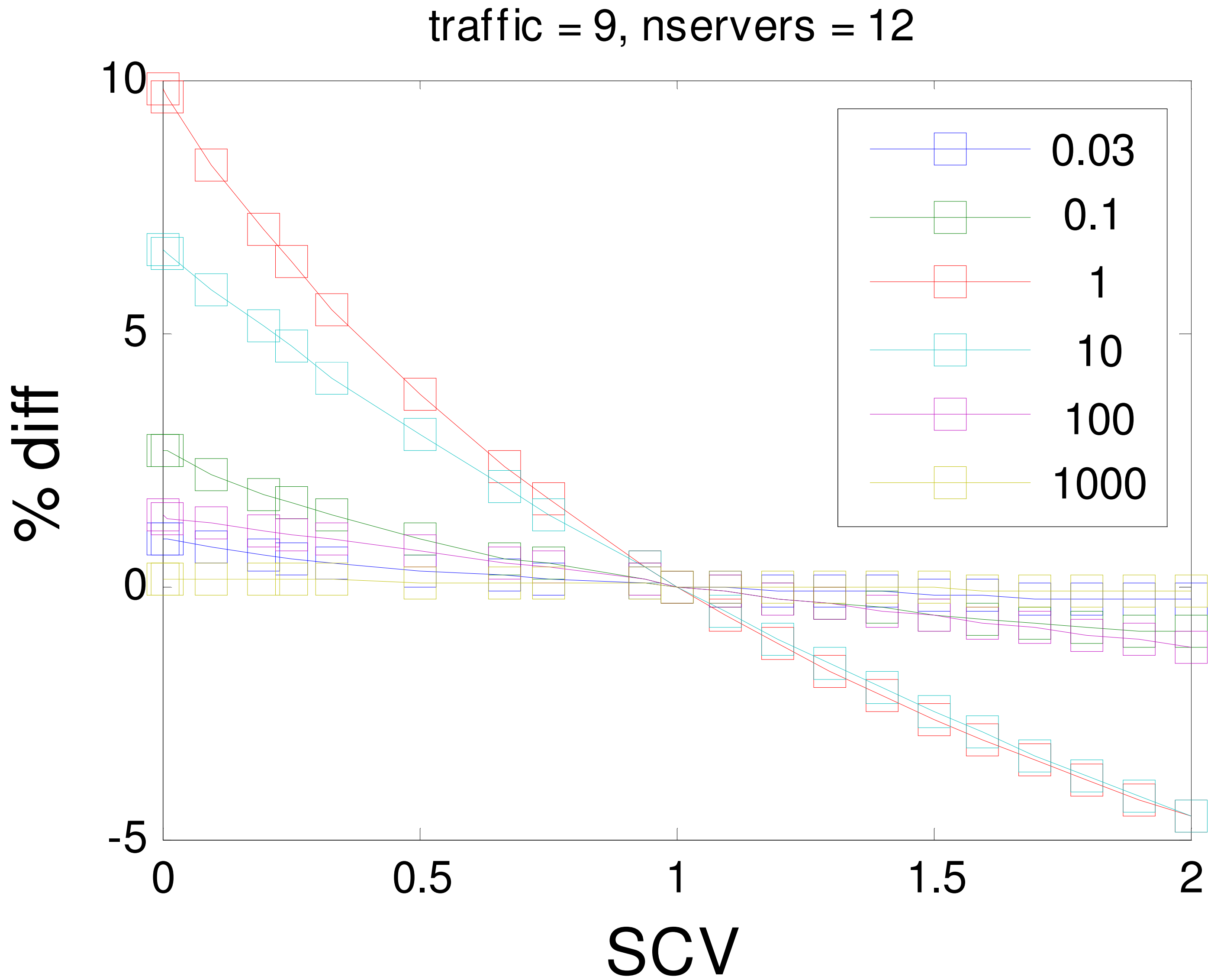
# Recall: % Diff in Pr(delay)



# NP: % Diff in Pr(delay)



# $H_2$*: % Diff in Pr(delay)



# Conclusions

- Do not use exponential retrials as an approximation to G-retrials when $CV < 0.1$ and retrial rate $<= 0.1$
- NP and $H_2^*$ distributions do not replicate simulations at low CV

# Queue-and-eh?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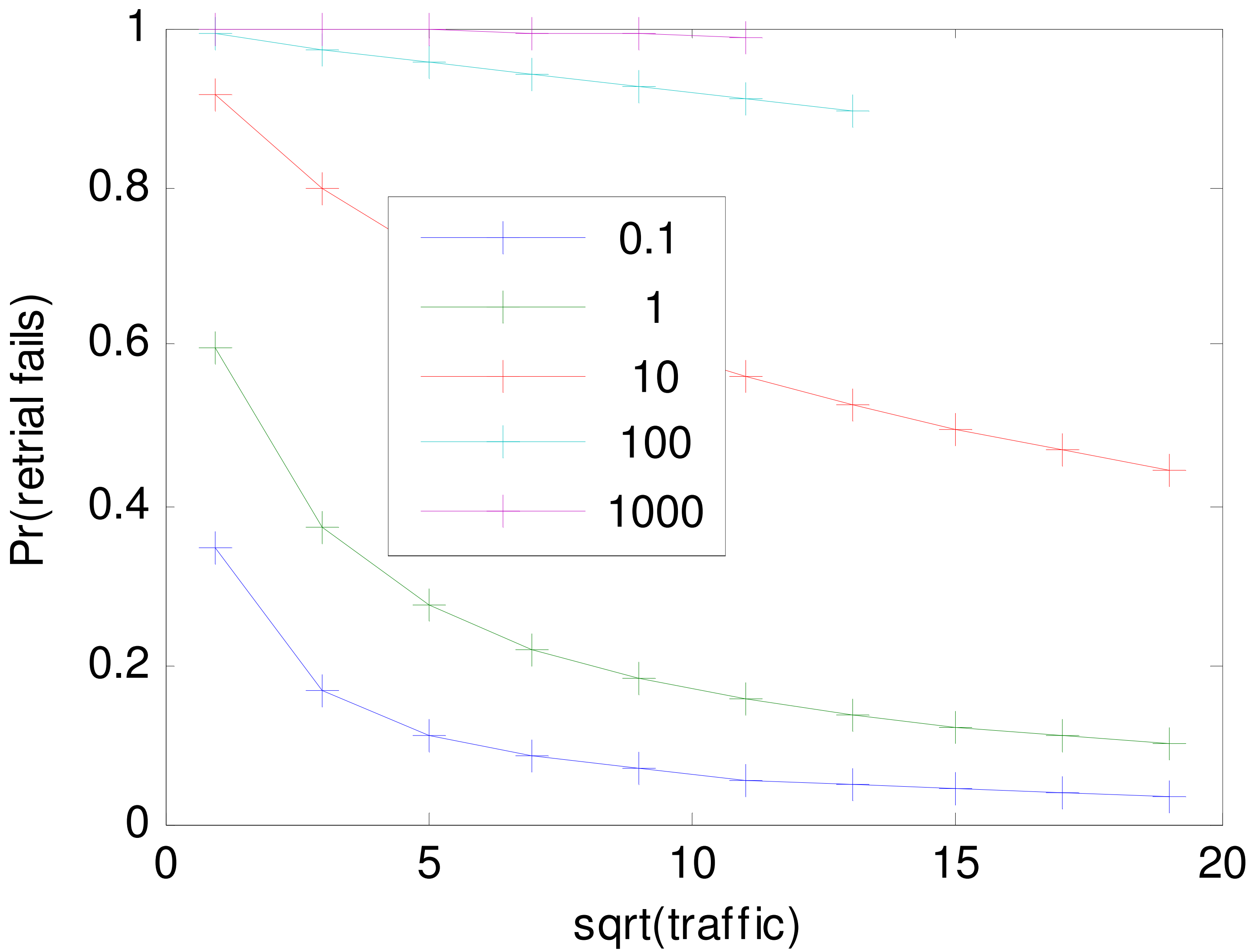
# Appendix

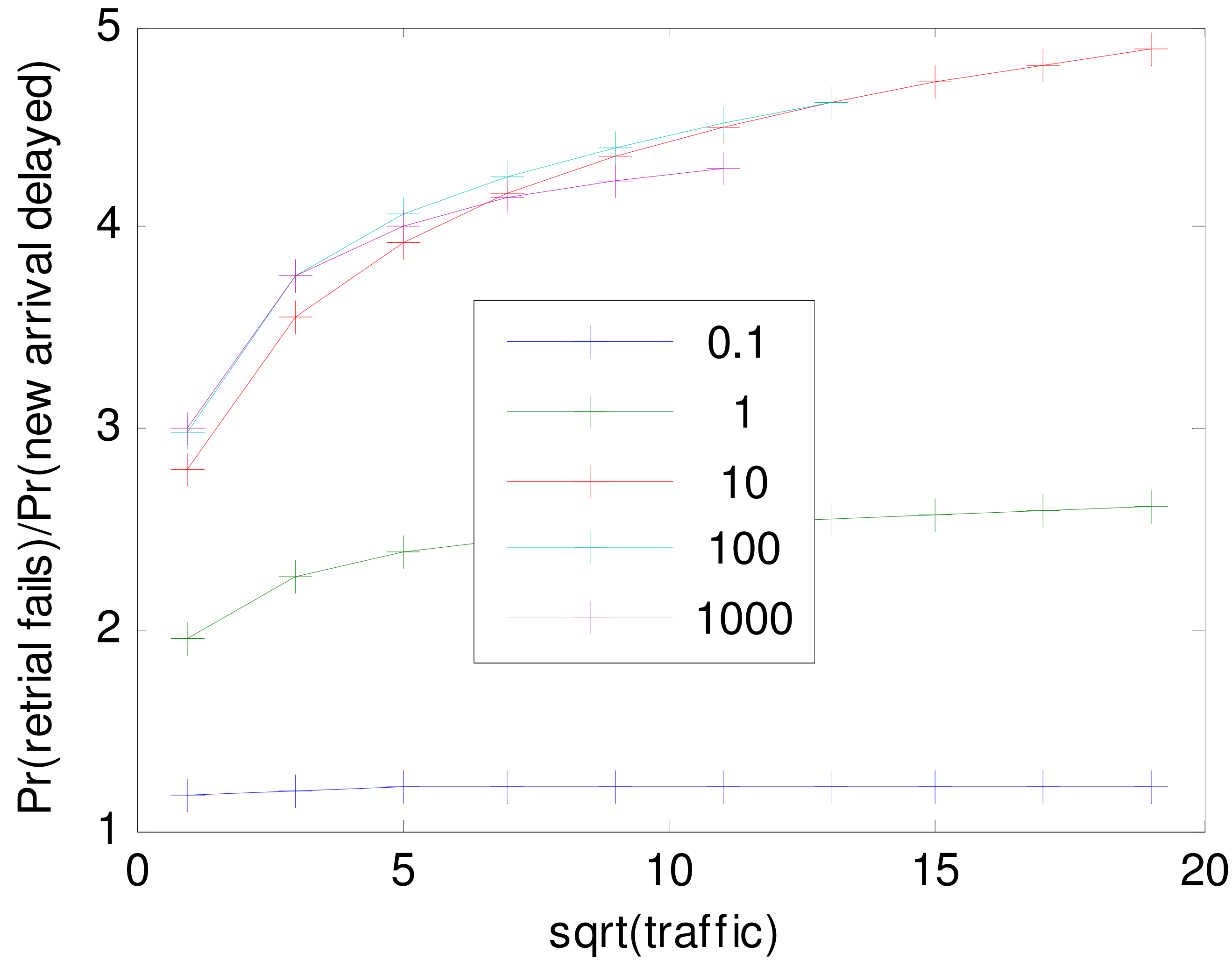
# General-Retrials literature

- Yang, Posner, Templeton, Li (1994): An approximation for M/G/1+G-retrials
- Many authors: only one person in orbit may retry (“constant retrial policy”)

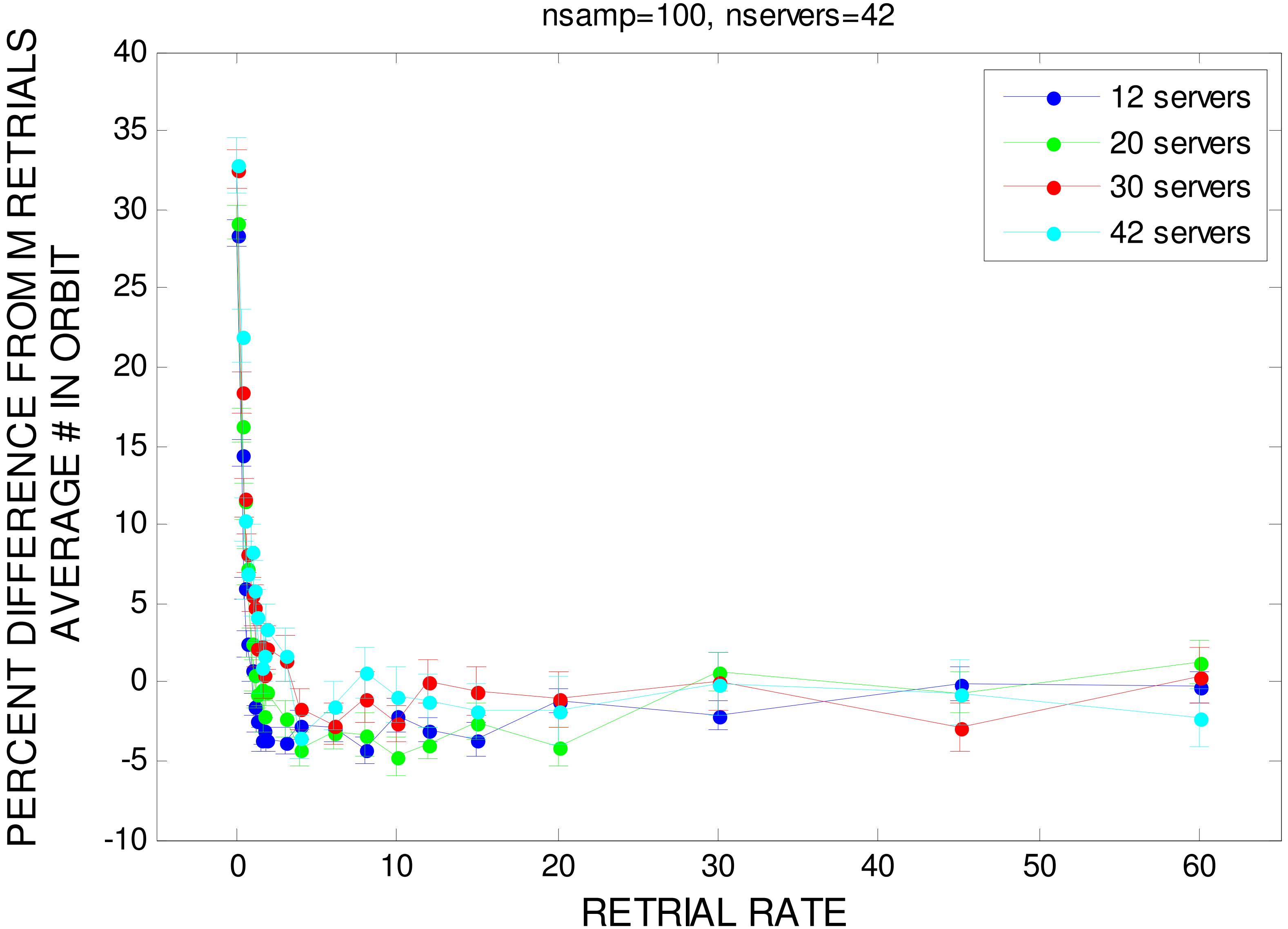
# Pr(retry fails) as system grows



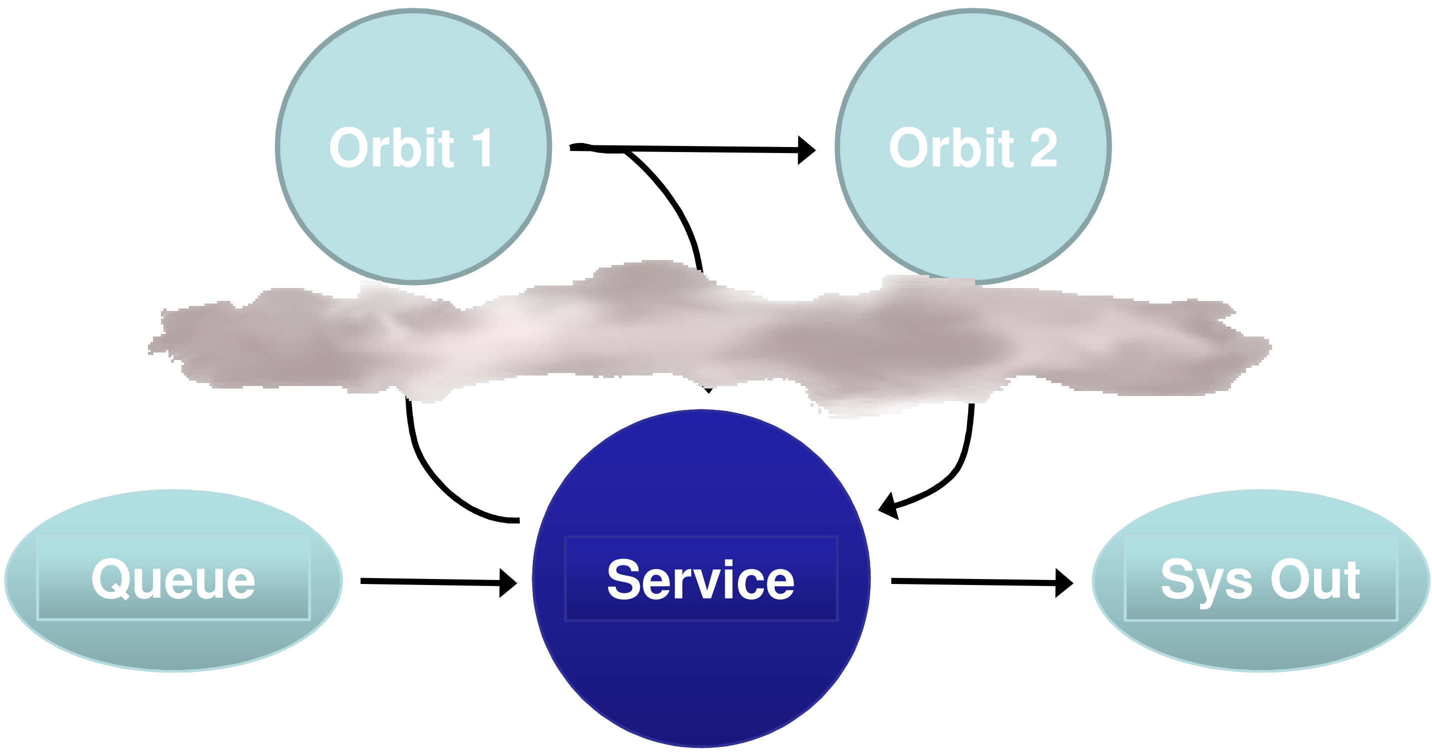
# P(retry fail)/P(new fail)



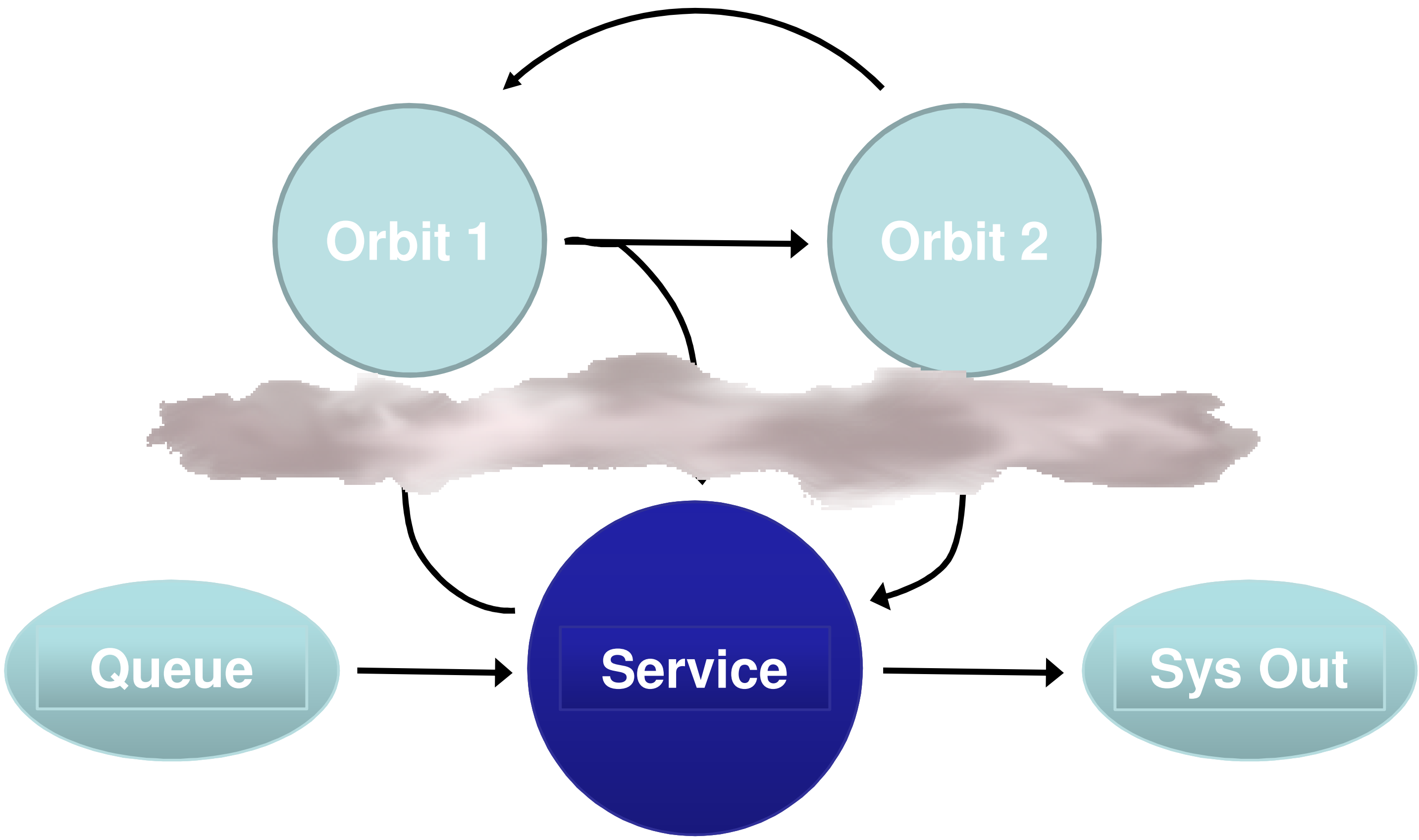
# % Diff: Average Number in Orbit



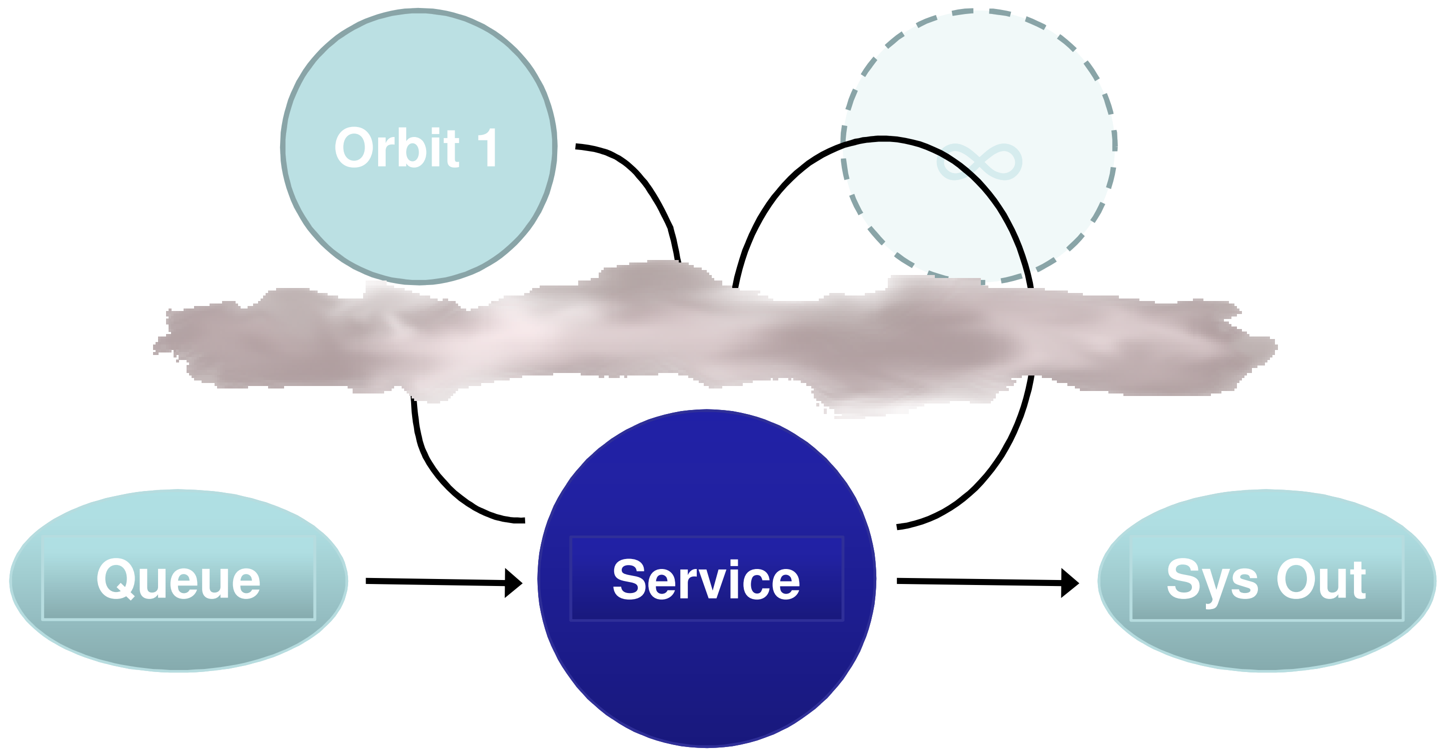
# Cox-Marie



# NP distribution



# $H_2$* distribution



# Very Low Retrial Rates, D-retrials

| RetryRate | Lo | StdErr | Expon. Lo | %diff from Exp | StdErr of % |
|---|---|---|---|---|---|
| 0.001 | 1931.987 | 3.332592 | 1414.9 | 36.54584 | 9.118938 |
| 0.01 | 194.7987 | 0.832853 | 142.39 | 36.80644 | 2.26279 |

| RetryRate | P(delay) | StdErr | Expon. Pd | %diff from Exp | StdErr of % |
|---|---|---|---|---|---|
| 0.001 | 0.149124 | 0.000172 | 0.13581 | 9.803055 | 0.001757 |
| 0.01 | 0.150076 | 0.000424 | 0.1362 | 10.1883 | 0.004161 |

# Very Low Retrial Rates, D-retrials

| RetryRate | Lo | StdErr | Expon. Lo | %diff from Exp | StdErr of % |
|---|---|---|---|---|---|
| 0.001 | 1932 | 3.3 | 1415 | 36.5 | 9.1 |
| 0.01 | 195 | 0.8 | 142 | 36.8 | 2.3 |

| RetryRate | P(delay) | StdErr | Expon. Pd | %diff from Exp | StdErr of % |
|---|---|---|---|---|---|
| 0.001 | 0.1491 | 0.0001 | 0.1358 | 9.8 | 0.002 |
| 0.01 | 0.1500 | 0.0004 | 0.1362 | 10.1 | 0.004 |