1 Problem Statement

The new Provost of Aluacha Balaclava College has hired Team 137, Inc. to create a new pay system for the faculty. The new system is to accommodate four types of teachers: Instructors, Assistant Professors, Associate Professors, and Full Professors. Teachers should be rewarded according to their rank and years of experience, with substantial salary increases for promotions.

Faculty without Ph.D. degrees are hired as instructors and are automatically promoted to Assistant Professors when they receive their degrees. Faculty with Ph.D. degrees are hired as Assistant Professors. Associate Professors may not be promoted before they have served seven years as Associates.

Faculty are paid periodically during the ten month school year, September through June. Raises are always effective beginning in September. The total amount of money available for raises varies, and is not known until March.

The new pay scheme should take into account increases in the cost of living, and the amount of money available for raises each year. Also, a transition plan should be provided to move the current faculty onto the new system, without ever decreasing anyone's salary. It should be possible to credit teachers with up to seven years experience gained prior to joining the college.

Any possible pay plan is subject to the following:

1.1 Constraints

1. If there is enough money for raises, then everyone gets a raise.
2. Teachers who are promoted according to the usual schedule of seven years as an Assistant and seven years as an Associate and who work 25 years or more should receive at retirement twice as much as a new Assistant Professor's salary.
3. Although there should be a reward for years of experience, the salaries of two teachers with equal rank should approach each other as they gain experience.
4. The salary for a newly promoted teacher should be about what it would have been in seven years, without the promotion.

2 Assumptions

Although payments are actually made throughout the school year, and salary decisions are made in March, we will assume that the decisions are made between discrete yearly salary payments.

We also assume that when the decisions are made, the Provost will have the budget for next year and an estimate of the cost of living increase. However, no information is available for years beyond the one for which salaries are being decided.

Since we are prohibited from decreasing anyone's salary when moving the current faculty to the new scheme, we will assume that there will always be enough money to pay everyone's salaries from the previous year. That is, there might be no money for raises, but we can at least pay the faculty at last year's nominal level.

We must give everyone a raise if anyone gets a raise, but we will assume that we can give unequal raises. Otherwise, we would just split the money evenly among the faculty, and the current pay system would not change very much at all.

Although Constraint 2 mentions 25 years and retirement, teachers are not forced to retire either at 25 years of experience or at 65 years of age (as the old law used to have it). However, we will assume an upper limit of 60 on the number of years of experience.

Constraints 2 and 4 refer to real dollar values. Otherwise, it would be extremely difficult to guarantee promising new Ph.D.'s that they will retire with twice their current salary. Besides, doubling the nominal salary in 25 years won't even keep up with 3% inflation.

Constraints 1 and 3 refer to nominal dollar values. The college always deals in nominal amounts when figuring budgets. There might not be enough to give the full cost-of-living increase; this would amount to a decrease in real salary.

Teachers who take longer to be promoted than is usual may receive as much of a raise or more if and when they finally do get promoted. That is, there is no penalty for being promoted late, other than the lost wages during that time.

New faculty will only be hired if there is enough money to pay their starting salary, including any years of previous experience they have. They will start out at that salary, and their salary will not be included in the amount available for raises. This means we do not have to give them a "raise" from \$0 to \$32,000 for their first year.

Retiring faculty take all their money with them; their salaries do not get thrown into the pool for raises.

Since no information was given about the transition from Assistant to Associate, we will assume that one must have seven years of experience total (not necessarily with this college, or as an Assistant) in order to be promoted.

3 Analysis of the Problem

It is currently late winter of 1995; it is too late for next year's salaries to be decided by the model proposed in this paper. The first year that will be on the new salary model is 1997. Even then, salaries will gradually move toward the target curves, since some teachers are being overpaid and cannot have their salaries reduced.

One solution to this is to set the target salaries so high that everyone needs a raise in order to attain the target. This is clearly not a solution that the college would favor, although it would be rather popular with the faculty.

It is difficult to conceive of a pay scheme that had no view of the future, yet managed to reliably satisfy the constraints regarding promotions and retirement. Therefore, the model should contain an overall view of how much a teacher at a certain rank with a certain number of years of experience should be paid.

This is certain to cause some friction between the college and the faculty; how much is a full professor with 10 years of experience worth? Moreover, at what rate should salaries increase, within the framework of the constraints? Should the model be set up to encourage or discourage retirement? Can someone be hired as a full professor with no experience? These are political questions for the Provost and faculty to negotiate; the model must handle whatever answers that negotiation produces. The one thing that the teachers must become accustomed to is that there is a certain value that their Provost places on them.

4 Design of the Model

We have a model that presents an overall goal for a pay system, and then adapts it to the real world. There are a pair of core systems (Logarithmic and Linear) for the administration and faculty to decide between. These cores are what the college would pay the faculty if it had enough money to do so, and some faculty would not mind getting less pay raise. These cores are also in real dollars, not adjusted to inflation. They are then adjusted for inflation (both historical and predicted) and adjusted to meet a finite budget. There are then two options to take care of teachers that, according to the new system, are being overpaid. Finally, the unlikely event of a budget excess is dealt with.

4.1 Variables

Let t be the current year; $t+1$ is the year for which salaries are being computed. Let t_i be the year that teacher i started at the college, and adjusted by the number of years of experience credited to that teacher upon entrance into the plan. Thus, if teacher i joined the college in 1994 with 4 years of experience, then $t_i = 1990$.

Let $T(i,t)$ denote the amount in real dollars that person i should get paid in year t, their Target for that year. This will depend on the rank of i and on the number of years of experience i has.

In the cores that follow, a_0, b_0, c_0, d_0 denote the initial salary of a Full Professor, Associate, Assistant, and Instructor, respectively. This is the amount paid to a teacher with no years of experience entering the plan. According to the problem statement, nobody can be hired as anything more than an Assistant; however, the model could easily handle such an occurrence. Indeed, one needs to estimate the "initial" salaries of Associates and Full Professors in order to start the pay system.

The initial salaries are of some concern. They are currently $d_0 = \$27{,}000$ for an instructor with no experience, and $c_0 =$\$32,000 for a new Assistant Professor with no experience. We must have $a_0 < 2c_0$, or else Full Professors would have a decreasing salary in order to hit $2c_0$ at 25 years. Convention forces $d_0 < c_0 < b_0 < a_0$. It will turn out (see Appendix A) that $a_0 = \$40{,}000$ and $b_0 = \$36{,}000$ are good estimates.

4.2 The Two Cores

According to Constraint 3, teachers of equal rank but different experience should have their salaries approach each other as time goes on. This leaves two possibilities: either the absolute difference of their salaries goes to zero (Logarithmic core), or the proportion of their salaries goes to one (Linear core).

One might expect a model to have a core that has a horizontal asympote, so a teacher's salary has a clear upper bound. However, as careers are limited to 60 years (Professor Methusela does not work at Aluacha Balaclava College), all salaries are bounded. As long as the proper rate constants are chosen, no teacher's salary will get too large for the college to handle.

The first core, the logarithmic, increases more rapidly at the beginning of someone's career than at the end. Larger raises occur early, but by the time one has gained 25 years of experience, the salary curve has really flattenned out. Here is the logarithmic core:

$$T(i,t) = \begin{cases} d_0 \log_{10}(d(t-t_i)+10) & \text{if } i \text{ is an instructor} \\ c_0 \log_{10}(c(t-t_i)+10) & \text{if } i \text{ is an assistant} \\ b_0 \log_{10}(b(t-t_i)+10) & \text{if } i \text{ is an associate} \\ a_0 \log_{10}(a(t-t_i)+10) & \text{if } i \text{ is a full professor} \end{cases}$$

The $+10$ term inside the logarithm allows the teacher's starting salary to be the coefficient of the logarithm expression. Indeed, the equation becomes (taking c as an example) $c_0 \log_{10}(0+10) = c_0 \cdot 1$, so the factors out front are the initial salaries, with no scaling necessary.

Linear Core:

$$T(i,t) = \begin{cases} c_0 + c(t-t_i-7) & \text{if } i \text{ is an Instructor} \\ c_0 + c(t-t_i) & \text{if } i \text{ is an Assistant} \\ b_0 + b(t-t_i) & \text{if } i \text{ is an Associate} \\ a_0 + a(t-t_i) & \text{if } i \text{ is a Full Professor} \end{cases}$$

The variables a,b,c,d are determined by constraints 2 and 4, outlined above. This guarantees that a Professor retiring at 25 years after the usual promotions will make twice as much (in real dollars) as a new Ph.D. entering as an Assistant. It also guarantees the equivalent of a seven year raise for someone who gets promoted. The calculation of these coefficients is a matter of applying constraints 2 and 4; they depend only on the initial salaries. See Appendix A for a brief derivation.

Note that in the linear core the Instructor salary is a seven year time shift of the of the Assistant salary. This results from the uncertainty of when an Instructor will receive her Ph.D. and become an Assistant. There would be no possible way to solve for a rate d explicitly. In response to this outcry, Team 137

Inc. decided to shift the Assistant salary curve to use for an Instructor salary curve. Team 137 Inc. found that a seven-year time shift would fit the given Instructor starting salary, $27,000, and Assistant starting salary, $32,000.

In Figures 4.2 and 4.2 we have graphed the ideal salaries for all ranks of faculty. Note how the logarithmic core tapers off after 25 years, giving more experienced teachers less and less of a raise each year, while the linear core keeps giving them the same raise. It is in this way that the logarithmic core encourages retirement.

4.3 The Real World

4.3.1 Inflation

Let $\gamma(t+1)$ be the true cost-of-living factor going from year t to year $t+1$; a typical value would be 1.03 for 3% inflation. Each person's real target salary for year t will be multiplied by the accumulated cost of living increases to produce the nominal target salary $N(i, t+1)$. If the new plan started in year t^0, then the nominal target salary is

$$N(i, t+1) = T(i, t+1)\hat{\gamma}(t+1) \prod_{j=t^0}^{t} \gamma(j)$$

where $\hat{\gamma}(t+1)$ is an estimate of the cost-of-living factor going from the current year to the next.

4.3.2 Finite Budgets

Of course, the college will not always have enough money to give each teacher the full raise each year. This calls for some way to portion out the raises in accordance with who deserves them the most. If two teachers each made $40,000 in 1996, and (according to one of the pay plans) the first has a target $41,000 in 1997 while the second's target is only $40,100, then the first should get a larger raise.

Let $n(i, t)$ be the amount in nominal dollars that teacher i gets paid in year t. Since budgets are limited, this will be at most $N(i, t)$. We have only some small amount of money, $M_r(t+1)$, for raises next year. This number is given to us by the outside world (the college's treasury). Let $M_n(t+1)$ be the amount needed next year for raises if all targets are to be met. That is,

$$M_n(t+1) = \sum_i \left(N(i, t+1) - n(i, t)\right),$$

where all M's are measured in nominal dollars. Usually, $M_n > M_r$; there isn't enough to give the faculty the raise they deserve

A fair way, then, to give raises is to give each person a raise in proportion to how much one is needed, where the proportion is the amount available for

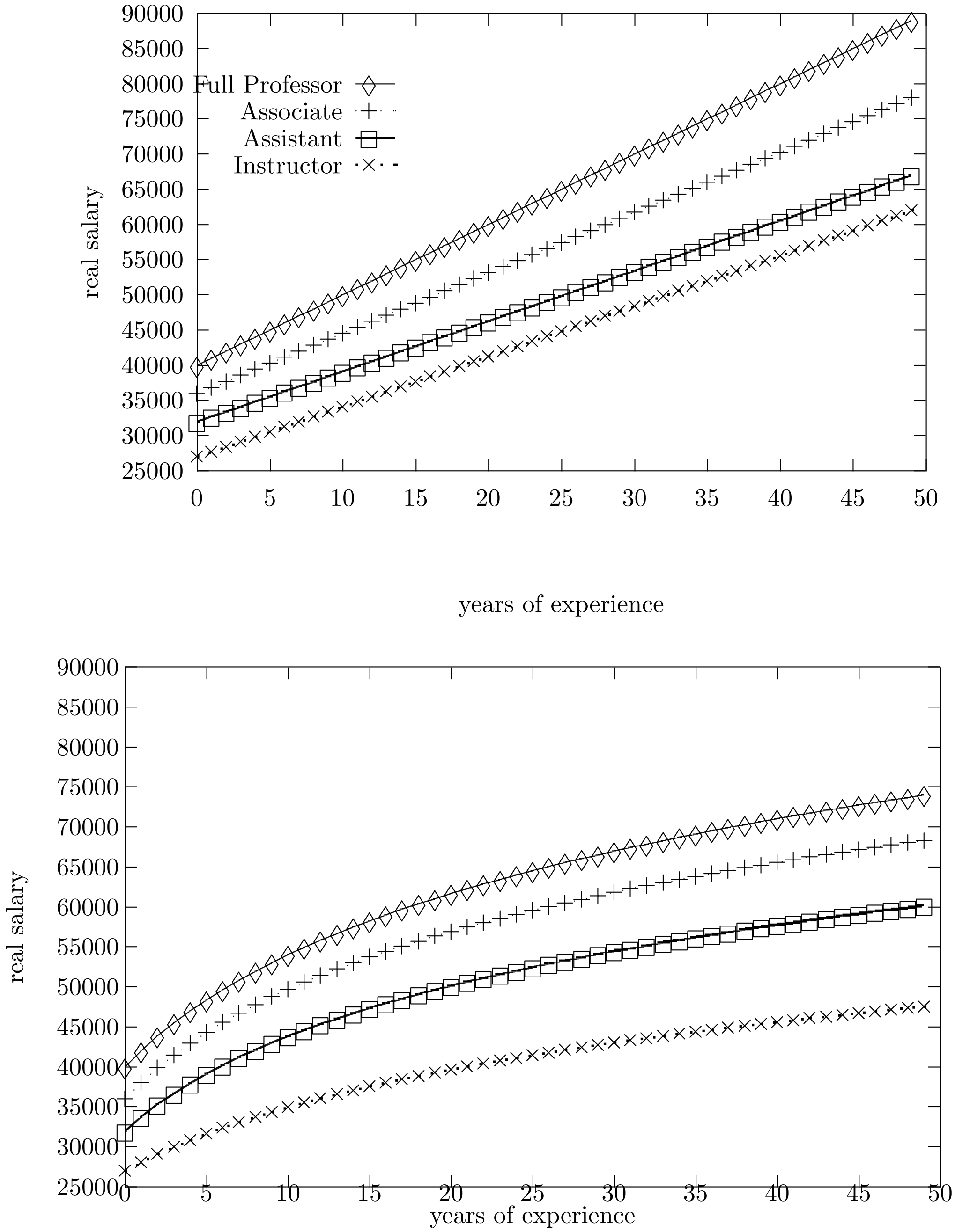

Figure 1: The ideal salaries using the logarithmic core

raises over the amount needed for raises. Thus, next year's salary for teacher i is

$$n(i,t+1) = n(i,t) + \text{raise}(i,t)$$

where i's raise from year t to year $t+1$ is

$$\text{raise}(i,t) = \left(\frac{M_r(t+1)}{M_n(t+1)}\right)(N(i,t+1) - n(i,t))$$

In this manner, teachers who are getting paid far less than their target will get a larger portion of the raises, bringing them closer to their target. See Figure 4.3.2 for a picture of how a teacher's salary would increase toward the target.

4.3.3 Bruised Egos

The college already has a faculty, and the former pay plan (if it may be called anything implying organization) has given some teachers more money than they deserve under the new system. Thus, some teachers' salaries need to be reduced. However, we can't actually cut salaries; indeed, if there is money available, everyone has to get a raise. However, one of our assumptions says that the raises can be unequal. To make things simple, we could give the overpaid teachers an ϵ-dollar raise each year until their target salary catches up with their actual salary. However, this is likely to bruise a few (overpaid) egos.

A good way to placate the overpaid would be to give them a new nominal target $O(i,t)$ that corresponds only to the projected cost of living increase:

$$O(i,t+1) = \hat{\gamma}(t+1)n(i,t)$$

We then treat the overpaid who are underneath their new target O just like those who are underneath their original target N. If there is no positive inflation that year, just give the overpaid teachers some small amount each, to make sure they get a raise. Now, recompute the amount of money needed for raises of both types, and portion out the money we have according to who needs it the most:

$$M_n(t+1) = \sum_i \begin{cases} N(i,t+1) - n(i,t) & \text{if } i \text{ would be underpaid} \\ O(i,t+1) - n(i,t) & \text{if } i \text{ would be overpaid} \end{cases}$$

The new salaries are then computed as follows:

$$n(i,t+1) = n(i,t) + \text{raise}(i,t)$$

where the raise from year t to year $t+1$ is

$$\text{raise}(i,t) = \left(\frac{M_r(t+1)}{M_n(t+1)}\right) \cdot \begin{cases} N(i,t+1) - n(i,t) & \text{if } i \text{ would be underpaid} \\ O(i,t+1) - n(i,t) & \text{if } i \text{ would be overpaid} \end{cases}$$

4.3.4 Excess Funds

Perhaps this belongs under an “Unreal World” section; but in the off chance that there is more money available than is needed to put everyone on target, there are a number of options available:

- *Raise everyone’s salary.* This has a negative consequence: it could put people over their targets for the next year, if the excess is very large. It could put the college into dire financial straits in the future, when the teachers’ salaries cannot be cut. However, if the excess is not very large, teachers will still be below target for the year after the excess, and not much harm is done.

- *Give everyone bonuses.* This would take care of the excess without raising the faculty’s expectations for years to come. This is a better option from the college’s point of view than raising salaries, and it is a common practice in industrial settings.

- *Give it to the General Fund,* perhaps for caffeine grants for sleep-starved students.

5 Model Verification

We have projected the performance of the proposed models over the next fifty years. We analyzed the model both with and without such influences as limited budgets, inflation, hiring of new faculty, promotion of faculty members, and retirement. We have also analyzed the effect on this performance of changes in the chosen constants a_0 and b_0 (see Section 4.1).

Figure 5 shows the long term effects of our model with the logarithmic core on the existing faculty. This is assuming no hiring, promoting, or retiring of faculty. This graph also ignores cost of living increases and possible monetary constraints.

Figure 5 shows the long term effects of our model with the linear core on the existing faculty. The graph makes the same assumptions as in Figure 5. From these two figures, we can see the our model will move the faculty toward a uniform salary system over time. Teachers whose current salaries are below the target specified by the model are given raises bringing them up to target. Teachers whose current salaries are above the target are held to a constant salary in real terms (though not in nominal terms, see Figures 5 and 6) until the target catches up with them.

Figure 5 shows how our model, with the logarithmic core, behaves in the presence of three percent annual inflation. In this graph, the faculty retire according to the schedule described earlier, which explains why the graph becomes more sparse at the left side. This graph assumes that the college has unlimited funds.

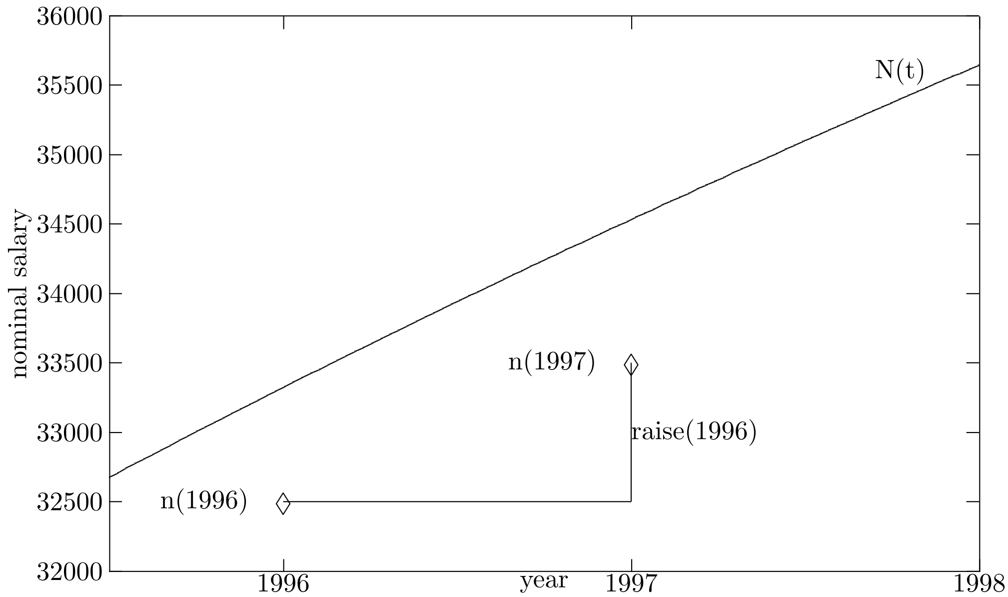

Figure 2: The effect of finite budgets on real salaries

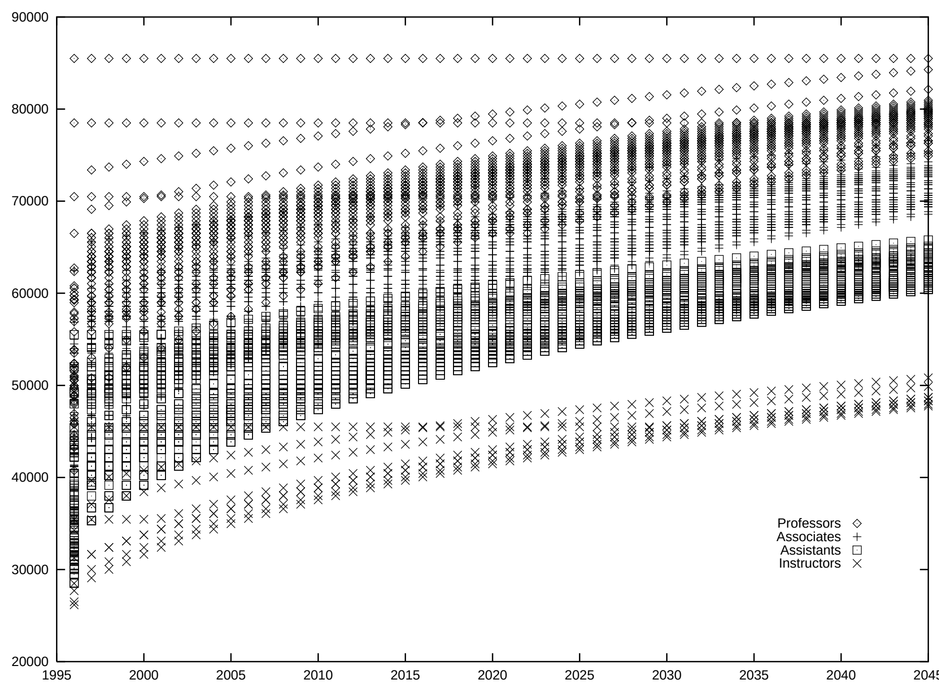

Figure 3: Long Term Transition with Logarithmic Core

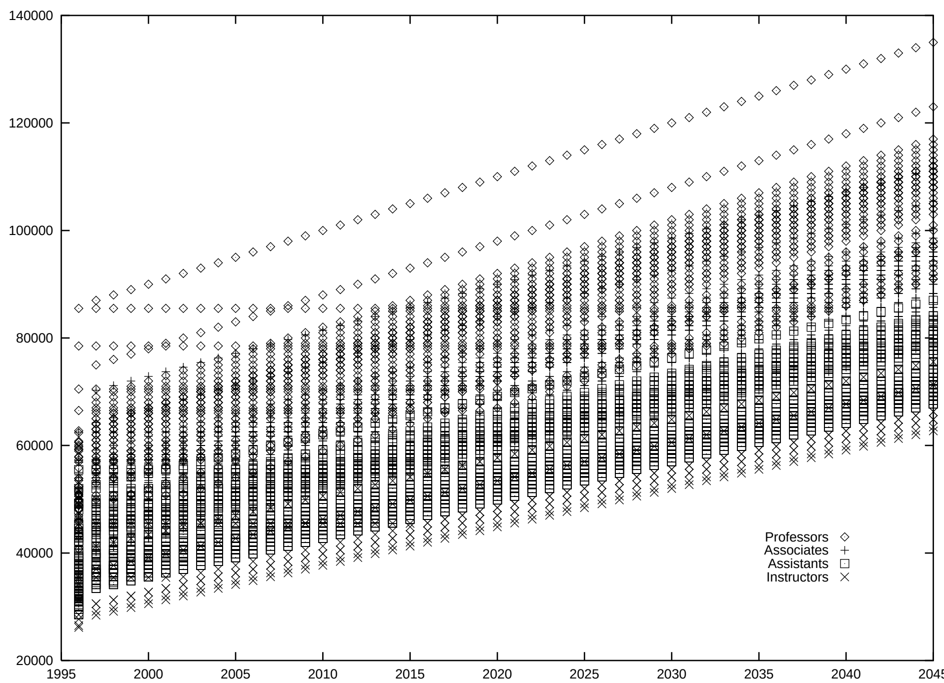

Figure 4: Long Term Transition with Linear Core

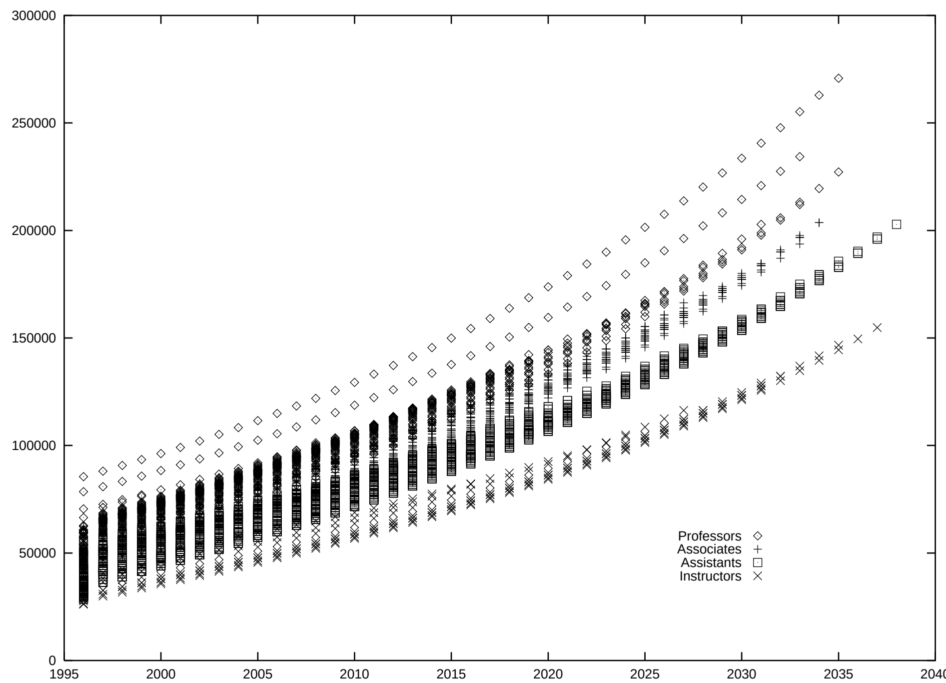

Figure 5: Logarithmic Core with Retirement and Inflation

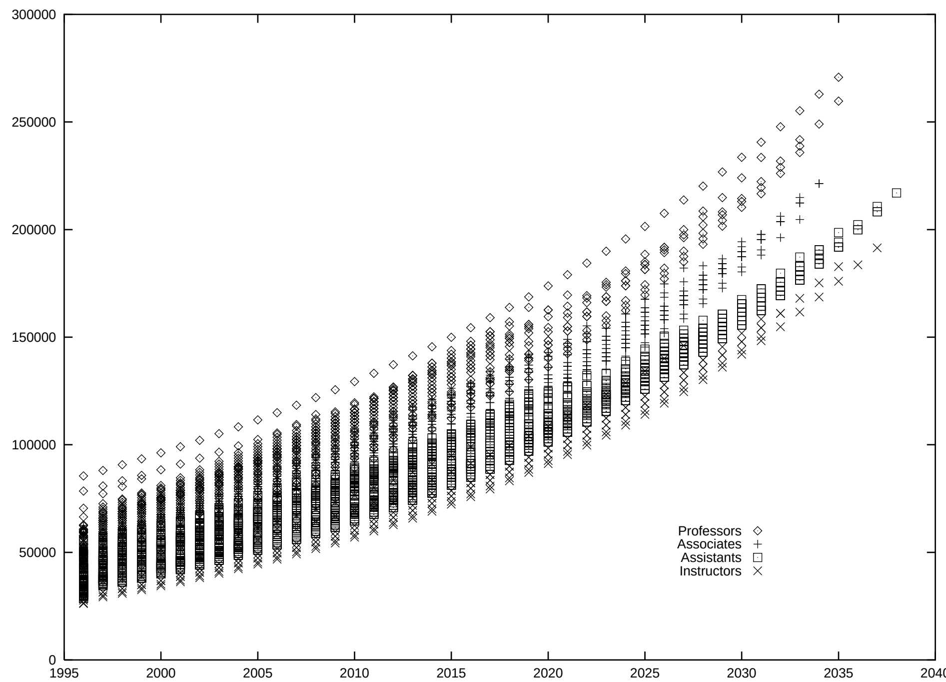

Figure 6: Linear Core with Retirement and Inflation

Figure 6 shows the behavior of the model with the linear core in the presence of the same conditions as in Figure 5. Note that these two graphs are essentially identical for large t, when the inflation terms dominates the model.

Figure 7 shows the behavior of the mode, with the logarithmic core, when faculty are promoted, eventually retire, and are replaced by new hires who also are promoted and eventually retire. As before, the model is able to bring the faculty into a coherent salary structure over time.

Figure 8 shows the behavior of the linear core under the same conditions as in Figure 7. Once again, the long term result is a uniform scale of faculty salaries.

Figure 9 shows the effect of budgetary constraints on the salary of a single professor working under the logarithmic core over time. This graph does not include promotions of the professor. Note that early differences between the professor's actual and target salary are eliminated as the professor's yearly raises get smaller.

Figure 10 shows the effect of budgetary constraints on the salary of a single professor working under the linear core over time. This graph does not include promotions of the professor. Note that a small difference between the actual and target salaries at the beginning of the simulation gets magnified at the end because the yearly raise never decreases under the linear model, but there is never enough money to give the professor a full year's raise. These graphs demonstrate the model's ability to cope with limited money situations.

5.1 Sensitivity Analysis

To analyze the sensitivity of our model to changes in a_0 and b_0, we varied these constants and examined the effect on the salaries of professors at each rank with fifty years of experience. We held c_0 and d_0 constant because they were provided in the problem statement. Tables 1 and 2 show our results.

Table 1 shows that in spite of ten percent variation in a_0 and b_0, the fifty year salaries of all four levels of faculty fluctuated by at most two percent under the logarithmic core.

Table 2 shows the fluctuation in salary is higher with the linear core, as is to be expected. It is still only eight percent, though. We conclude from this that our model, especially with the logarithmic core, is relatively insensitive to its initial parameters.

5.2 Long term performance

In making long term predictions, many real-world influences on our model had to be simulated. Specifically, we had to decide how many professors were going to change status each year. This includes new hires, promotions, and retirements. Also, we had to simulate the money available for promotion, and arrive at a reasonable cost of living factor for each year. We accomplished all of this with the aid of probability.

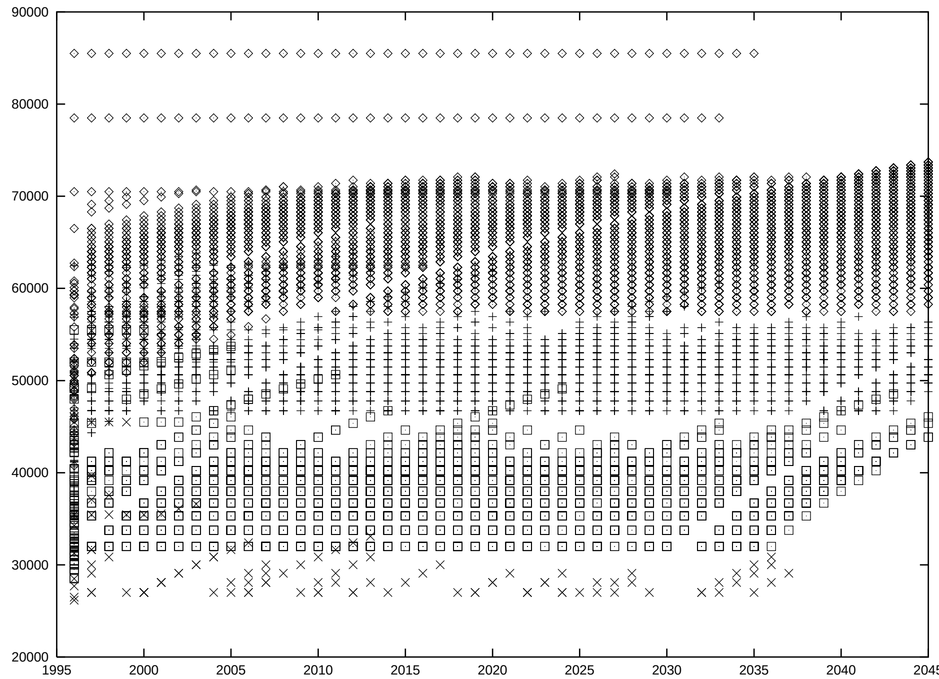

Figure 7: Hiring, Promotion, and Retirement - Logarithmic

a_0	b_0	Professor	Associate	Assistant	Instructor
40000	36000	74106.92	68311.86	60179.86	47555.53
42000	36000	73996.20	68544.99	60365.24	47601.24
40000	38000	74016.92	68306.33	60898.46	47744.38
42000	38000	73996.20	68548.13	61061.04	47788.07
38000	34000	73938.22	68048.43	59529.56	47390.58

Table 1: Effect of variations in initial conditions on salary after fifty years of employment under the logarithmic core

a_0	b_0	Professor	Associate	Assistant	Instructor
40000	36000	89000.00	78000.00	67000.00	62000.00
42000	36000	86916.67	79944.44	67972.22	62833.33
40000	38000	89000.00	75333.33	71666.67	66000.00
42000	38000	86916.67	77277.78	72638.89	66833.33
38000	34000	91083.33	78722.22	61361.11	57166.67

Table 2: Effect of variations in initial conditions on salary after fifty years of employment under the linear core

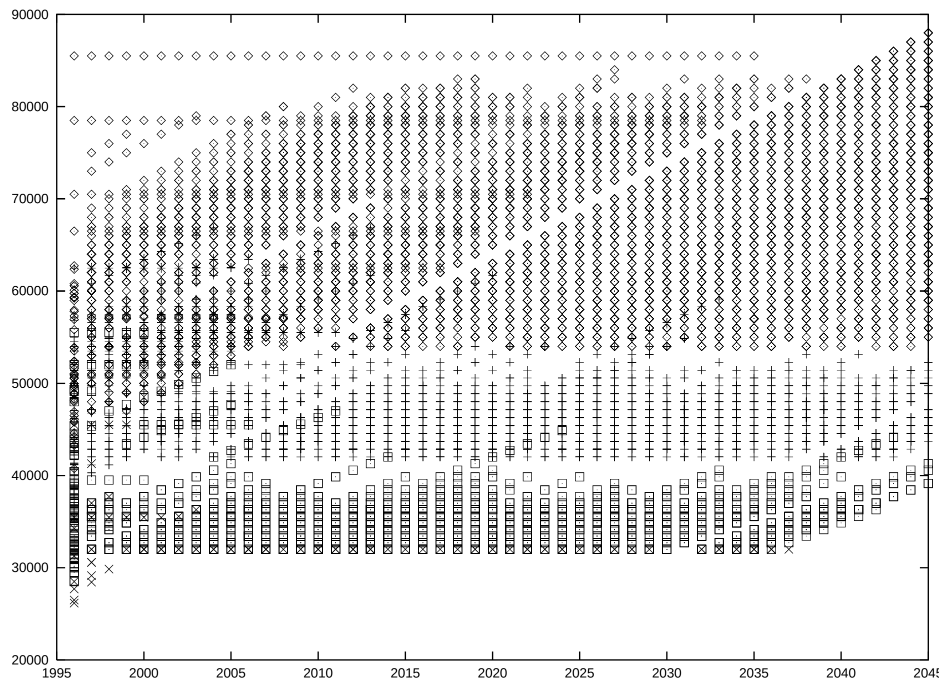

Figure 8: Hiring, Promotion, and Retirement - Linear

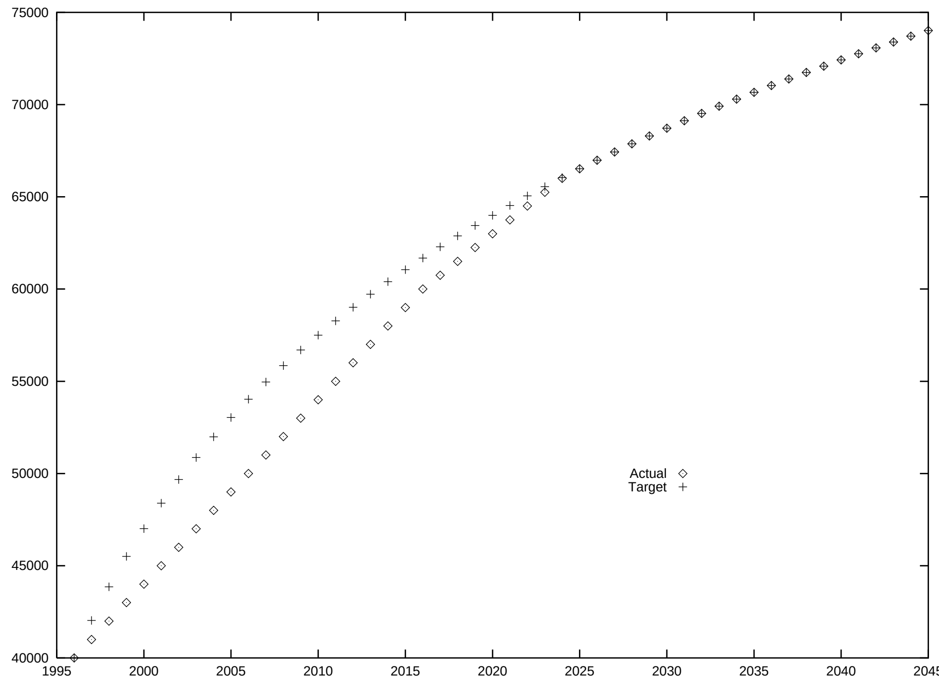

Figure 9: Logarithmic Core with Monetary Pressure

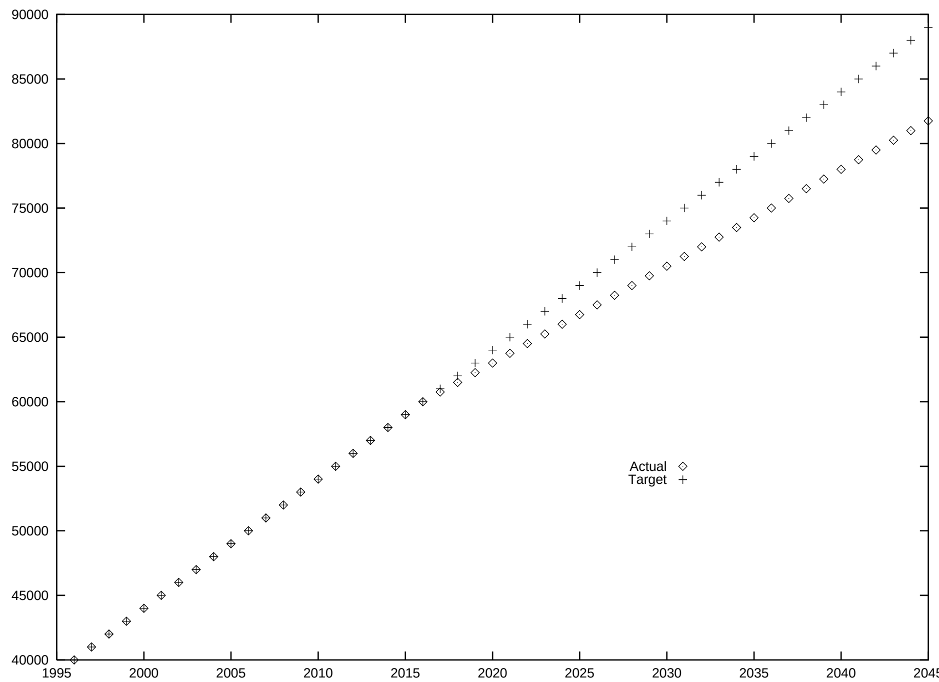

Figure 10: Linear Core with Monetary Pressure

By examining the initial data, we concluded that that the college hires a mean of 9 faculty per year with a standard deviation of 5 on a discritized normal distribution. We decided more or less arbitrarily that faculty will retire after working for 40 years, with a standard deviation of two years, again on a discritized normal distribution. For promotions, we decided that fifty percent of Assistant Professors would become Associate Professors in 7 years, twenty-five percent would become Associate Professors in 8 years, and so on. We realize this is not guaranteed to terminate in finite time, however in our case it did. We used the same probability distributions for Associate Professors being promoted to Full Professors.

As for cost of living increases, we used three percent inflation per year, or $\gamma_i = 1.03\ \forall i$. We also assumed that the college would always be able to accurately predict this value for the next year. Small fluctuations in any γ_i will have negligible affects on the model's performance, as will small inaccuracies in the college's yearly predictions. Finally, we analyzed the model in the presence of both limited and unlimited money supplies. For limited money, we made a constant amount of money available for raises and chose this constant such that it would be inadequate soon after the initial year of the simulation.

6 Strengths and Weaknesses

The logarithmic option encourages teachers to retire earlier than the linear model, since the logarithm curve flattens out at higher values. This means that the raise for a professor with 40 years of experience is small relative to the raise for someone with 26 years of experience. This gives the Provost a way to encourage or discourage retirement, based on the needs of the college.

If the college wishes to adjust the real values of the starting salaries, everyone's salaries will change, not just those hired after the change is made.

The faculty must be willing to settle for a higher potential salary, instead of a guaranteed higher salary, for promotions and retirement. As long as the college can't guarantee enough money for everyone's raises, this will remain.

Salary increases were calculated according to a professor on a track with the minimum years between promotions. This causes late promotions to receive the equivalent of an eight- or nine-year raise at their current rank. For example, a teacher promoted in eight years will receive a raise greater than the raise awarded for a promotion in seven years. Furthermore, all future promotions (if any) will have larger raises. Likewise, a professor retiring at 26 years experience, instead of 25 years will receive a salary greater than twice the salary of a first year assistant. (See Appendix A)

7 Appendix A

The two cores allow the initial salaries to be set by the outside world; however, the rate at which pay increases depends on these initial salaries. The constraints apply in this manner:

Constraint 2: Consider a new Ph.D. that is hired with no prior experience. If promotions occur on time, then years 0 through 6 are spent as an Assistant, years 7 through 14 as an Associate, and years 15 and on are as a Full Professor. Thus, the Ph.D.'s first year as an Associate is the 7th year; it should correspond in pay to the 15th year as an Assistant:

$$T(\text{Associate}, 7) = T(\text{Assistant}, 14)$$

Similarly, the Ph.D. becomes a Full Professor at the 15th year; that should correspond to the twenty-first year as an Associate:

$$T(\text{professor}, 15) = T(\text{associate}, 21)$$

We don't have to track inflation and budget constraints through the actual year because we are dealing in real dollars, and the salary curves don't change: ideally, Associate X with 15 years of experience in 1997 makes the same real amount as Associate Y with 15 years of experience in 2010.

Constraint 4: Similar to the previous constraint:

$$2T(\text{Assistant}, 0) = T(\text{Professor}, 25)$$

Since one's salary increases with time, those who retire after their 25th year receive more than twice $T(\text{assistant}, 0)$, which fits the constraint.

Earlier we discussed the reasoning behind choosing the Instructor salary curve in the linear core as a time shift. We also mentioned that there is no explicit method of solving for a linear rate constant for the Instructor salary (See The Two Cores). We run into the same problem for the logarithmic model, except a time shift would not produce an accurate result for the given starting salaries of an Associate and Instructor. Thus, we needed to choose an average year to promote the Instructors to Assistants. Since an Instructor at least one year as an Instructor, we needed to choose a year greater than zero. However, an Instructor may be promoted after exactly one year; if we based the salary curve on anything greater than this, these Instructors would receive less than a seven-year raise upon promotion. Therefore, we choose year one to be our Instructor promotion year.

Substitution and simplifying produce the following rate constants:

7.1 Logarithmic Core:

$$\begin{aligned}
a &= \tfrac{1}{24} \quad (10^{2c_0/a_0} - 10) \\
b &= \tfrac{1}{21} \quad ((14a + 10)^{a_0/b_0} - 10) \\
c &= \tfrac{1}{14} \quad ((7b + 10)^{b_0/c_0} - 10) \\
d &= \tfrac{1}{8} \quad ((c + 10)^{c_0/d_0} - 10)
\end{aligned}$$

7.2 Linear Core:

$$\begin{aligned} a &= \tfrac{1}{24} \quad (2c_0 - a_0) \\ b &= \tfrac{1}{21} \quad (14a + a_0 - b_0) \\ c &= \tfrac{1}{14} \quad (7b + b_0 - c_0) \end{aligned}$$

How do the starting salaries $a_0 = 40,000$ and $b_0 = 36,000$ work out so well? For the linear core, look at the initial salary for Instructors: $c(0 - 7) + c_0 = -7c + c_0$. c is determined by b and b_0; b is determined by a and a_0. So, a_0 and b_0 affect the Instructors' initial salary. With a_0 and b_0 chosen as they were, the starting instructor's salary comes out to \$27,000, which is exactly what was specified in the problem statement.