

Renewal Theory bits and pieces

Andrew M. Ross

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1 Basics

We will call the times between renewals X_1, X_2, X_3 etc., and these variables are IID with mean $E[X]$ and standard deviation σ . It is convenient to introduce the “Coefficient of Variation”, $c = \sigma/E[X]$, which is often expressed as a percent. For example, if the average time between renewals is 15 minutes and the standard deviation is 5 minutes, then $c = 5/15 = 1/3 \approx 33\%$. The advantage of this is that if we switch to expressing time in hours, the c does not change; it is still 33 percent. We also use the Squared Coefficient of Variation, $c^2 = (\sigma/E[X])^2$, because it appears in quite a few formulas. It is usually *not* expressed as a percent. It is important to keep in mind that Exponential distributions have $c = 1 = c^2$. If a distribution has $c^2 < 1$, it is less variable than Exponential, such as Erlang distributions, which are sums of IID Exponentials. If a distribution has $c^2 > 1$, it is more variable than Exponential; the classic example is Hyperexponential distributions, which are mixtures of different Exponentials.

Other important definitions are: the time of the n 'th renewal,

$$S_n = \sum_{i=1}^n X_i$$

and the number of renewals from time 0 to time t , not including the renewal at time 0:

$$N(t) = \text{number of renewals in } (0, t]$$

Also important is the “Renewal Function”, which is denoted $m(t)$ and is the mean number of renewals by time t : $m(t) = E[N(t)]$. According to Nelson (page 258), its Laplace transform is

$$\tilde{m}(s) = \frac{\tilde{F}(s)}{s(1 - \tilde{F}(s))}$$

The renewal function is often hard to compute. Figure 1 shows the renewal function for 3 processes: one with Hyperexponential-2 lifetimes, one with Exponential lifetimes (that is, a Poisson process), and one with Erlang-100 lifetimes. Notice that $m(t)$ doesn't really approach the diagonal line for the non-Poisson processes. We will get to this later.

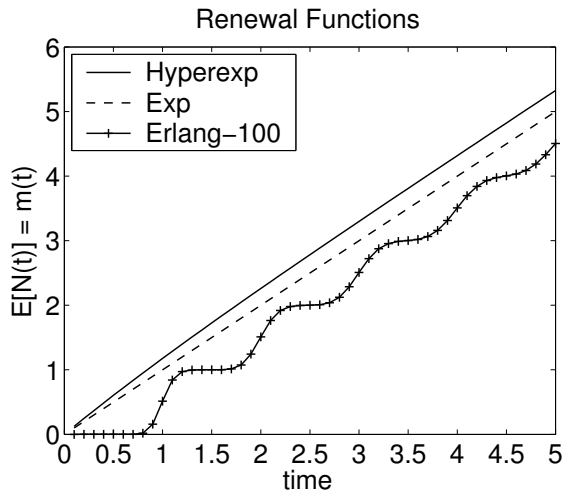


Figure 1: Renewal functions for Hyperexponential-2, Exponential, and Erlang-100 lifetime distributions

2 Central Limit Theorem, and Adjustments

The classic Central Limit Theorem applies easily to the time of the n 'th renewal, for very large n :

$$S_n \sim \text{Normal}(nE[X], n\sigma^2)$$

In practice, $n = 30$ is probably large enough if $c^2 \leq 1$.

The Central Limit Theorem for Renewal Processes (CLTRP) is slightly more complicated. Instead of working on S_n , it works on $N(t)$ for large t . The basic version is

$$N(t) \sim \text{Normal}\left(\frac{t}{E[X]}, \frac{t}{E[X]}c^2\right)$$

This often appears in books in the following not-so-nice form:

$$N(t) \sim \text{Normal}\left(\frac{t}{E[X]}, \frac{t\sigma^2}{E[X]^3}\right)$$

We might be tempted to think the variance should be $\sigma^2 t/E[X]$, since we expect $t/E[X]$ events, each of which has variance σ^2 . However, this formula does not even have the right dimensions; $N(t)$ and $\text{Var}(N(t))$ are dimensionless, whereas $\sigma^2 t/E[X]$ has the dimensions of time-squared. How big should t be in order to use the CLTRP? Big enough so $t/E[X] > 30$, or so, if $c^2 < 1$.

Now, it turns out that the mean and variance listed above are not as simple as they might appear. We will step back for a while and consider two functions: $y = 2x$ and $y = 2x - .5$. You will agree, I hope, that these two functions never

get closer as x grows large. However, they get closer *relative* to their size. That is,

$$\lim_{x \rightarrow \infty} \frac{2x}{x} = 2$$

and

$$\lim_{x \rightarrow \infty} \frac{2x - .5}{x} = 2$$

The above statement of the CLTRP is ignoring constant terms in the mean and the variance, because it is thinking about relative distances. Here are the formulas with the offset terms included:

$$E[N(t)] = m(t) \rightarrow \frac{t}{E[X]} + \frac{c^2 - 1}{2}$$

For the variance, define $k_1 = E[X]$, $k_2 = E[X^2] = \sigma^2 + E[X]^2$, and $k_3 = E[X^3]$ be the first 3 raw (not central) moments of the lifetimes.

$$\text{Var}(N(t)) \rightarrow \frac{t}{E[X]}c^2 + \frac{5k_2^2}{4E[X]^4} - \frac{2k_3}{3E[X]^3} - \frac{k_2}{2E[X]^2}$$

Notice that the offset for the mean includes the 2nd moment of lifetimes, and the offset for the variance includes the 3rd moment of lifetimes. Also notice that for Poisson processes, which have $c^2 = 1$, the mean offset is zero. The mean offset is negative for lifetimes which are less variable than Exponential, and positive for those which are more variable than Exponential. The most negative that the offset can be is $-1/2$, for deterministic lifetimes. The formula for the variance is not common, but occurs in Tijms' "Stochastic Models" page 75 Problem 1.6, and also in Cox, "Renewal Theory".

3 Inspection Paradox

The inspection paradox is this: if a renewal process has been going on for a long time and you arrive and ask what the lifetime of the current item is, the answer is on average longer than the average lifetime of all items. That is, if each item lasts a week on average, the randomly-inspected item will (on average) last longer than a week.

This paradox is related to the "Equilibrium distribution" of an item's lifetime, which is the same as the distribution of residual life. This is also the same as the distribution of the age of an item. The formula for this is

$$F_e(x) = \frac{1}{E[X]} \int_0^x (1 - F(y)) dy$$

Its density is then

$$f_e(x) = \frac{1}{E[X]} (1 - F(x))$$

Its Laplace transform is (from Ross' "Stochastic Processes" book)

$$\tilde{F}_e(s) = \frac{1}{E[X]} \cdot \frac{1 - \tilde{F}(s)}{s}$$

Its raw (not central) moments are (from Nelson, page 265)

$$E[X_e^k] = \frac{1}{E[X]} \cdot \frac{E[X^{k+1}]}{k+1}$$

So, in particular, its mean is

$$E[X_e] = \frac{1}{E[X]} \cdot \frac{E[X^2]}{2} = \frac{c^2 + 1}{2} E[X]$$

The mean lifetime of an inspected item is twice this: $(c^2 + 1)E[X]$.

Also, the failure rate of the equilibrium distribution is $1/\text{MRL}(t)$, where $\text{MRL}(t)$ is the mean residual lifetime at time t (of the original lifetime, not the new equilibrium lifetime).

It turns out that the age and excess of a renewal process at any particular time are not independent. So, if we want the distribution of the overall lifetime of the inspected item, we can't just take a convolution of the age distribution and the excess distribution (which are both the equilibrium distribution). As it turns out, we can apply a length-biased sampling argument and we find that the density of the lifetime of an inspected item is

$$f_i(x) = x \cdot f(x) / E[X]$$

where the subscript i denotes the inspected lifetime.

4 Other Applications of SCV

For a Compound Poisson process with rate λ , where each event has size Y_i , the mean of the total size by time t is of course $\lambda t E[Y_1]$. The variance you might guess to be $\lambda t \text{Var}(Y_1)$, but that intuition includes only the variability in the event sizes, and not the variability in the number of events. Altogether, after using the conditional variance formula, we get $\lambda t E[Y_1^2]$ which is not very intuitive. We can transform this to $\lambda t (c_Y^2 + 1) \cdot (E[Y_1])^2$, which is a little better.