Notes on Recursive FFT (Fast Fourier Transform) algorithm
Fall 2010, COSC 511
Susan Haynes

(1) The Fourier Transform transforms `a (|a| = n)` vector in spatial or time domain to a vector in frequency domain.

(2) The Fourier Transform is invertible.

(3) Convolution (e.g., polynomial multiplication) is $O(n^2)$.

(4) Convolution in spatial domain is the same as pair-wise multiplication in the frequency domain.

(5) Fourier transform of a vector `a` can be performed by a matrix - vector multiply, where the matrix encodes the Fourier transform and the vector is `a`. Matrix - vector multiply is $O(n^2)$.

(6) The FFT uses a particular matrix, F, where each element is one of the n-th roots of 1 (unity). The element of the matrix at row i, column j (starting at (0,0)) is ω_n^{i*j} . To compute `FFT(a)` using the matrix `F` and vector a is $O(n^2)$ (see comment 5). FFT^{-1} element at (i, j) is (1/n) ω_n^{-i*j}

(7) ω_n^i has some nice properties:

- $\omega_n^i * \omega_n^j$ computes to one of the n complex n-th roots.
- $\omega_n^{i\%n} = \omega_n^i$ adding any multiple of n to the exponent, gives the original ω_n^i)
- $\omega_{dn}^{dk} = \omega_n^k$ (e.g., $\omega_8^6 = \omega_4^3$) AKA cancellation property
- If n = 0 is even, then the squares of the n nth-roots of unity are the n/2 complex (n/2)-th roots of unity. E.g., Consider the 4 fourth roots of unity: ω_4^1, ω_4^2, ω_4^3, ω_4^4. Square them: ω_4^2, ω_4^4, ω_4^6, ω_4^8. By the previous cancellation property. The squared roots are ω_2^1, ω_2^2, ω_2^3, ω_2^4. By the mod property, those squared roots are ω_2^1, ω_2^0, ω_2^1, ω_2^0 which are exactly the two 2nd roots of unity.

(8) The FFT can be implemented as a divide-and-conquer (hence a recursive) algorithm, giving O(n lg n). The analysis is similar to the analysis for merge-sort.

(9) Convolution $p1 \otimes p2 = q$, using FFT is O(n lg n) (rather than $O(n^2)$, see comment 3) because:

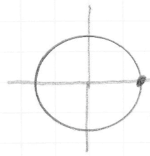
P1 = FFT(p1) is O(n lg n)
P2 = FFT(p2) is O(n lg n)
Q = P1 * P2 (pairwise) is O(n)
q = FFT^{-1}(Q) is O(n lg n)

(10) The pseudocode for FFT (recursive version) from Cormen, Leisserson, et al, is:

```
FFT (a)   {              // 1D array a = (a_0, a_1, … , a_{n-1})
    n = length (a)           // n is a power of 2
    if (n == 1)
        return a
    ω_n = e^{2πi/n}              // ω_n^1
    ω = 1
    a^[0] = (a_0, a_2, a_4, … a_{n-2} )
    a^[1] = (a_1, a_3, a_5, … a_{n-1} )
    y^[0] = FFT(a^[0])
    y^[1] = FFT(a^[1])
    for (k=0; k <= n/2-1; k++) {
        y_k = y_k^[0] + ω y_k^[1]
        y_{k+(n/2)} = y_k^[0] - ω y_k^[1]
        w = ω w_n
        }
    return y                 // y is vector, |y| = n
    }
```

(11) Unit circle with roots, FFT, and FFT^{-1} are given on the following pages.

$n = 1$

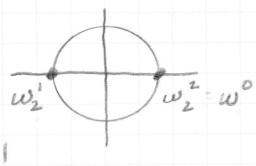


$\omega_1^1 = \omega_1^0 = e^{i0} = 1$

FFT: $\begin{bmatrix} \omega^{0 \cdot 0} \end{bmatrix} = \begin{bmatrix} \omega^0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

$FFT^{-1} = \begin{bmatrix} \omega^{-0 \cdot 0} \end{bmatrix} = \begin{bmatrix} \omega^0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

$n = 2$

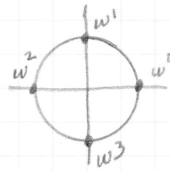


$\omega^0 = +1$
$\omega^1 = -1$

FFT: $\begin{bmatrix} \omega^{0 \cdot 0} & \omega^{0 \cdot 1} \\ \omega^{1 \cdot 0} & \omega^{1 \cdot 1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$FFT^{-1} = \begin{bmatrix} \omega^0 & \omega^0 \\ \omega^0 & \omega^{-1 \cdot 1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$n = 4$



$$w^1 = e^{i\frac{\pi}{2}} = i$$

$$w^2 = e^{i\pi} = -1$$

$$w^3 = e^{i\frac{3\pi}{2}} = -i$$

$$w^4 = w^0 = 1$$

$$FFT = \begin{bmatrix} w^{0\cdot 0} & w^{0\cdot 1} & w^{0\cdot 2} & w^{0\cdot 3} \\ w^{1\cdot 0} & w^{1\cdot 1} & w^{1\cdot 2} & w^{1\cdot 3} \\ w^{2\cdot 0} & w^{2\cdot 1} & w^{2\cdot 2} & w^{2\cdot 3} \\ w^{3\cdot 0} & w^{3\cdot 1} & w^{3\cdot 2} & w^{3\cdot 3} \end{bmatrix}$$

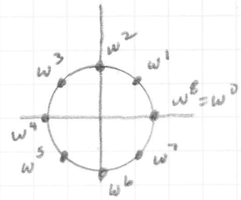
$$= \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$FFT^{-1} = \frac{1}{4}\begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^{-1} & w^{-2} & w^{-3} \\ w^0 & w^{-2} & w^{-4} & w^{-6} \\ w^0 & w^{-3} & w^{-6} & w^{-9} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i^{-1} & 1 & -i^{-1} \\ 1 & -1 & 1 & -1 \\ 1 & -i^{-1} & -1 & i^{-1} \end{bmatrix}$$

$n = 8$



$$FFT = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^0 & w^2 & w^4 & w^6 & w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^1 & w^4 & w^7 & w^2 & w^5 \\ w^0 & w^4 & w^0 & w^4 & w^0 & w^4 & w^0 & w^4 \\ w^0 & w^5 & w^2 & w^7 & w^4 & w^1 & w^6 & w^3 \\ w^0 & w^6 & w^4 & w^2 & w^0 & w^6 & w^4 & w^2 \\ w^0 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w^1 \end{bmatrix}$$

$* \; w^1 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$

$w^2 = e^{i\frac{\pi}{2}} = i$

$w^3 = e^{i\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{-1+i}{\sqrt{2}}$

$w^4 = e^{i\pi} = -1$

$w^5 = e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \frac{-1-i}{\sqrt{2}}$

$w^6 = e^{i\frac{6\pi}{4}} = -i$

$w^7 = e^{i\frac{7\pi}{4}} = +\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \frac{1-i}{\sqrt{2}}$

$w^8 = e^{i2\pi} = w^0 = 1$

$$FFT^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{\sqrt{2}}{1+i} & \frac{1}{i} & \frac{\sqrt{2}}{-1+i} & -1 & \frac{\sqrt{2}}{-1-i} & \frac{-1}{i} & \frac{\sqrt{2}}{1-i} \\ 1 & \frac{1}{i} & -1 & -\frac{1}{i} & 1 & \frac{1}{i} & -1 & -\frac{1}{i} \\ 1 & \frac{\sqrt{2}}{-1+i} & -\frac{1}{i} & \frac{\sqrt{2}}{1+i} & -1 & \frac{\sqrt{2}}{1-i} & \frac{1}{i} & \frac{\sqrt{2}}{-1-i} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{\sqrt{2}}{-1-i} & \frac{1}{i} & \frac{\sqrt{2}}{1-i} & -1 & \frac{\sqrt{2}}{1+i} & -\frac{1}{i} & \frac{\sqrt{2}}{-1+i} \\ 1 & -\frac{1}{i} & -1 & \frac{1}{i} & 1 & -\frac{1}{i} & -1 & \frac{1}{i} \\ 1 & \frac{\sqrt{2}}{1-i} & -\frac{1}{i} & \frac{\sqrt{2}}{-1-i} & -1 & \frac{\sqrt{2}}{-1+i} & \frac{1}{i} & \frac{\sqrt{2}}{1+i} \end{bmatrix}$$

* calculate the rectangular representation for w^1, then use symmetry for w^3, w^5, w^7

Pythagorean theorem says
$r^2 = x^2 + y^2$

for w^1 $\quad 1^2 = x^2 + x^2$

$1 = \sqrt{2x^2}$

$= \sqrt{2}\, x$

$\frac{1}{\sqrt{2}} = x$

$\frac{1}{\sqrt{2}} = y$